Making the square-root formula compatible with capital allocation*

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Abstract

Modern regulatory capital standards, such as the Solvency II standard formula, employ a correlation-based “square-root formula” for aggregating, for example, market risks, default risks and insurance risks. To support business steering, companies will allocate the aggregate capital requirement back to business units and risk drivers. We demonstrate that the classical concepts of correlation can cause a substantial bias between the capital allocation based on the square root formula and based on the true multivariate risk distribution. As an alternative, we propose so-called “partial-derivative implied” correlations and show that they make the square-root formula unbiased in terms of the aggregate capital requirement as well as its first and second-order derivatives with respect to exposures. We exemplify that these conditions are needed to correctly identify optimal portfolios: the classical types of correlation can induce a suboptimal portfolio and a drop in the company’s safety level clearly below the desired level.

JEL classification: G22, G28, G32.

Keywords: Capital allocation, Risk aggregation, Solvency II.

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1 Introduction

Risk diversification is at the core of an insurer’s business model. Hence, regulatory capital requirements need to appropriately account for risk dependencies and diversification between different risk. Several modern regulatory capital standards, such as the Solvency II standard formula, employ a correlation-based approach for risk aggregation, which determines the aggregate capital requirement of the company in two steps.\(^1\) Firstly, the stand-alone capital requirements for \(n\) univariate risks are determined, resulting in a capital requirement vector \(x \in \mathbb{R}^n\). Secondly, an \(n \times n\)-matrix \(R\) of correlation parameters is used to determine the aggregate capital requirement, \(\text{CapR}\), by the square-root formula:

\[
\text{CapR} = \sqrt{x^T R x}
\]  

(1)

This formula is accurate with \(R\) being the Pearson correlation matrix if risks follow an elliptical distribution and the capital requirement is determined by a positive-homogeneous, translation-invariant and law-invariant risk measure, such as the Value-at-Risk (VaR) (cf. McNeil et al., 2015, pp. 295). For elliptically distributed risks, the capital requirement determined by (1) helps to prevent the regulated company from moral hazard: any change in its portfolio is accompanied by an adjustment in the capital requirement such that the safety level—i.e. the confidence level of the risk measure—remains stable. Hence, the company cannot alter its safety level by shifting its portfolio in favor of shareholders and at the expense of policyholders.

\(^1\)BIS (2010) provide an overview of the risk aggregation approaches used in regulatory capital standards. More recently, the International Capital Standard (ICS) makes use of a correlation-based risk aggregation approach; cf. IAIS (2019, p. 246 f.).
In reality, risk profiles of insurance companies typically deviate from an elliptical distribution, due to heavy-tailed risks and non-linear dependencies between risks. Several studies indicate that the square-root formula in this context can cause or intensify unfavorable portfolio management decisions. Fischer and Schlütter (2015) demonstrate with a simulation study that the standard formula may encourage corner solutions of an insurer’s investment portfolio; hence, small changes in the standard formula’s parameters can substantially impact the preferred strategy. Braun et al. (2017) study a Markowitz portfolio problem in connection with the standard formula; they find that the standard formula does not accurately account for diversification effects, substantially restricts the set of feasible asset allocations and may induce an inefficient portfolio selection. Braun et al. (2018) demonstrate that the performance measure Return on Risk-Adjusted Capital (RORAC) in combination with the capital required by the standard formula may induce insurers to choose a weakly-diversified investment portfolio with high asset risk. Chen et al. (2019) provide empirical evidence for moral hazard induced by deficiencies in the risk aggregation approach of the risk-based capital (RBC) system in the United States. The RBC system employs the square-root formula under the assumption that the major risk categories are uncorrelated (i.e., the matrix \( R \) in (1) is an identity matrix). Chen et al. highlight that questionable marginal capital requirements resulting from this set-up have increased the insurers’ risk-taking in terms of their fixed-income securities portfolio.

This paper seeks after an approach to calibrate the square-root formula such that it provides a suitable basis for portfolio management decisions\(^2\) for other than elliptical distributions. Our target is to calibrate the square-root formula such that a company’s preferred portfolio does not depend on whether the capital requirement is determined

\(^2\)The term “portfolio” can be understood in a wide sense, including the asset portfolio or the portfolio of business segments. The latter is in the focus of the analyses of Myers and Read (2001) and Zanjani (2002).
based on the square-root formula or based on the true multivariate risk distribution. To
correctly identify an optimal portfolio, the square-root formula needs to reflect first and
second-order derivatives of the aggregate capital requirement with respect to exposure
changes in line with the true risk distribution.\(^3\) The Euler (synonymous: gradient) capi-
tal allocation coincides with those first-order derivatives. Hence, the square-root formula
particularly needs to provide the Euler capital allocation in line with the true distribu-
tion.\(^4\)

Classically, it is assumed that the matrix \(R\) in (1) has ones on its diagonal (as it is the
case for Pearson correlation matrices, for example). Section 3 of this article provides
some simple examples demonstrating that this assumption can make it impossible to
calibrate the square-root formula as a suitable basis for portfolio management decisions.
For \(n = 2\) risks, \(R\) has only one free parameter; in general, it is not possible to set this
parameter such that the Euler capital allocation to both risks coincide with the results
based on the true risk distribution. For \(n \geq 3\) risks, it is possible to calibrate \(R\) such that
the square-root formula assesses Euler capital allocations correctly, but then it does in
general not assess second-order derivatives correctly. The European insurance supervi-
sory authority (EIOPA 2014, p. 9) recommends calibrating the square-root formula with
so-called VaR-implied correlations, which have been initially proposed by Campbell et al.
(2002). The VaR-implied correlation parameters are determined such that the square-
root formula aggregates the capital requirement for two or more risks accurately to the
capital requirement based on the true multivariate distribution. The most straightfor-
ward implementation of EIOPA’s recommendation is to identify “pairwise” VaR-implied

\(^3\)As shown by Paulusch (2017), the square-root formula defines a differentiable and homogeneous risk
measure and hence it is possible to calculate those derivatives.
\(^4\)In connection with portfolio optimization based on a performance measure such as Economic Value
Added (EVA) or RORAC, the Euler allocation has been considered by Tasche (2008), Buch et al. (2011)
and Diers (2011).
correlations for each two risks (cf. Campbell et al., 2002); alternatively, Mittnik (2014) has proposed a joint identification for \( n \geq 3 \) risks. We demonstrate that VaR-implied correlations as well as Pearson correlations can lead to substantial biases in the Euler capital allocation.

To make the square-root formula a suitable basis for portfolio management decisions, we propose in section 4 to take a different view on the matrix \( R \). For elliptical distributions, the entries of \( R \) globally coincide with the second-order partial derivatives of the squared aggregate capital requirement with respect to changes in the capital requirement of the univariate risks. For general distributions (given some weak conditions about differentiability), these second-order partial derivatives uniquely define a symmetric matrix. We show that the square-root formula in connection with this “partial-derivative implied tail correlation matrix” is a useful local approximation: for the calibration portfolio, it provides the aggregate capital requirement, Euler capital allocations and second-order partial derivatives of the aggregate capital requirement in line with the results from the true risk distribution. If the distribution is not elliptical, the diagonal elements of the partial-derivative implied tail correlation matrix can deviate from one.

Section 5 provides an impact assessment for an insurance company that determines the EVA-optimal portfolio in terms of the volumes of its three lines-of-business; the capital requirement is defined as the 99.5% Value-at-Risk. We determine the “true” optimal portfolio which maximizes EVA if the capital requirement is determined based on the true multivariate risk distribution. We calibrate \( R \) with different methods and consider two situations. In the first situation, \( R \) is calibrated based on the true optimal portfolio. Even in this situation, Pearson and VaR-implied correlations induce the insurer to choose a suboptimal portfolio, since they misstate the impact of portfolio changes for
the aggregate capital requirement. The true VaR confidence level of the chosen portfolio can be substantially below the desired 99.5%. If $R$ is calibrated such that it implies the true Euler capital allocation, but with ones on the diagonal, it can imply that a saddle point is mistaken as a local EVA-optimum. Calibrating $R$ based on partial-derivative implied correlations overcomes these issues and the company correctly characterizes an optimal portfolio and a saddle point. In the second situation, the portfolio for calibrating $R$ deviates from the true optimal portfolio. We demonstrate that the steering signals obtained from the square-root formula in connection with partial-derivative implied correlations are relatively robust when changing the calibration portfolio; in connection with a classical calibration of correlations, the missteering impulses of the square-root formula intensify.

Overall, our article contributes to the literature about risk aggregation for non-elliptical distributions. Previous articles focus on the aggregate capital requirement and account for incomplete information. Besides Mittnik (2014), who deals with the calibration of $R$, Embrechts et al. (2013) and Bernard et al. (2018) provide thresholds for the aggregate capital requirement. Given that partial derivatives of the capital requirement with respect to exposure changes matter, our article demonstrates that the calibration of $R$ is challenging even with complete information about the multivariate risk distribution. The question of estimating the suggested partial-derivative implied correlations under partial information is left for future research. We will briefly get back to this question in our conclusion in section 6.
2 The square-root formula and capital allocation

Suppose an insurance company is confronted with \( n \) risks. The random variable \( X_i, i \in \{1, ..., n\} \), with \( \mathbb{E}(X_i) < \infty \) models the loss in the company’s equity capital due to risk \( i \). Let \( \varrho \) be a homogeneous, translation-invariant and law-invariant risk measure. The stand-alone capital requirement for risk \( i \) is defined in regard of the unexpected loss, \( X_i - \mathbb{E}(X_i) \), as

\[
x_i = \text{CapR}(X_i) = \varrho(X_i - \mathbb{E}(X_i)) = \varrho(X_i) - \mathbb{E}(X_i)
\]  

(2)

The vector \( x = (x_i)_{i=1}^{n} \in \mathbb{R}^n \) contains all \( n \) stand-alone capital requirements. The company’s aggregate loss is

\[
X = \sum_{i=1}^{n} X_i,
\]  

(3)

and the aggregate capital requirement is given by

\[
\text{CapR}(X) = \varrho(X - \mathbb{E}(X))
\]  

(4)

To derive risk-adjusted performance measures and support portfolio management decisions, it is reasonable to reallocate the capital requirement in (4) to the \( n \) risks. To this end, Tasche (2008) suggests to consider the function

\[
u = (u_1, \ldots, u_n)^T \mapsto f_X(u) = \text{CapR} \left( \sum_{i=1}^{n} u_i \cdot X_i \right)
\]  

(5)

The vector \( (u_1, \ldots, u_n)^T \) can be viewed as the company’s exposures to risks \( 1, \ldots, n \). Without loss of generality, we may assume that the \( X_i \) are scaled such that the coordinates
\( u = 1_n = (1, \ldots, 1)^T \) reflect the company’s actual portfolio with \( f_X(1_n) = \text{CapR}(X) \). Assuming that \( f_X(u) \) is differentiable in \( u \), Tasche (2008) suggests to define the capital allocated to risk \( k \) by the partial derivative of the aggregate capital requirement with respect to the exposure to risk \( k \):\(^5\)

\[
\frac{\partial}{\partial u_k} f_X(1_n) = \frac{d}{dh} \text{CapR}(X + h \cdot X_k) \bigg|_{h=0}
\]

(6)

We will denote the result of (6) as the marginal capital requirement of risk \( k \). Tasche (2008) shows that it is reasonable to calculate the RORAC of risk \( k \) based on the marginal capital requirement of that risk: if the RORAC of risk \( k \) increases the RORAC of the entire company, a small increase in the exposure to risk \( k \) enhances the RORAC of the company.

Given that the function in (5) is homogeneous of degree one, Euler’s theorem implies that

\[
\sum_{k=1}^{n} \frac{\partial}{\partial u_k} f_X(1_n) = \text{CapR}(X)
\]

(7)

Hence, the marginal capital requirements add up to the company’s aggregate capital requirement. Since (4) and (6) are based on the complete multivariate risk distribution, we will denote them later on as the “true” aggregate and “true” marginal capital requirement.

\(^5\)Similarly, Myers and Read (2001, pp. 557-559) propose to allocate capital to lines of insurance based on the impact of marginal exposure changes. In contrast to Tasche, who assumes a homogeneous risk measure, Myers and Read postulate that the marginal default values (the first-order derivative of the present value of the insurer’s option to default with respect to a line’s size) shall be equal across the lines of business.
The capital requirement defined by the square-root formula is denoted by

$$\sqrt{x^T R x} = \sqrt{\sum_{i,j=1}^{n} \varrho_{ij} x_i x_j}, \quad (8)$$

where $x_i$ is the stand-alone capital requirement of risk $i$ according to Eq. (2), and $R = (\varrho_{ij})_{i,j=1}^{n}$ is a matrix of correlation parameters. Using the Hadamard product $u \circ x = (u_1 x_1, \ldots, u_n x_n)^T \in \mathbb{R}^n$, the analogy to function $f_X$ in equation (5) in terms of the square-root formula is

$$u = (u_1, \ldots, u_n)^T \mapsto g_x(u) = \sqrt{(u \circ x)^T R (u \circ x)}, \quad (9)$$

where the subscript $x$ of $g_x$ indicates the dependence on the actual portfolio $x$ and must not be confused with a derivative. As shown by Paulusch (2017), $g_x$ is a homogeneous function in $u$. Hence, the marginal capital requirement of risk $k$ according to the square-root formula is given by

$$\frac{\partial}{\partial u_k} g_x(u) = \frac{\sum_{i=1}^{n} \varrho_{ki} u_i x_i}{\sqrt{(u \circ x)^T R (u \circ x)}} \cdot x_k, \quad (10)$$

with

$$\sum_{k=1}^{n} \frac{\partial}{\partial u_k} g_x(1_n) = \sqrt{x^T R x} \quad (11)$$

Analogously to the marginal capital requirement in (6), $\frac{\partial}{\partial u_k} g_x(1_n)$ measures the change in the aggregate capital requirement when the exposure to risk $k$ is marginally increased. In matrix notation and at $u = 1_n$, the vector of marginal capital requirements is determined as

$$\frac{(Rx) \circ x}{\sqrt{x^T R x}} \quad (12)$$
Finally, the second-order partial derivatives of the square-root formula \( g_x \) with respect to exposure changes are obtained as

\[
\frac{\partial^2}{\partial u_k \partial u_\ell} g_x(u) = -\sum_{i=1}^n \varrho_{ki} u_i x_k \cdot \sum_{j=1}^n \varrho_{\ell j} u_j x_\ell \frac{x_k x_\ell}{(u \circ x)^T R (u \circ x)^{1.5}} + \rho_{k,\ell} \frac{x_k x_\ell}{\sqrt{(u \circ x)^T R (u \circ x)}}
\]

with

\[
\frac{\partial^2}{\partial u_k \partial u_\ell} g_x(\mathbb{1}_n) = \left. \rho_{k,\ell} x_k x_\ell - \left( \frac{\partial g_x}{\partial u_k} \frac{\partial g_x}{\partial u_\ell} \right) \right|_{u=\mathbb{1}_n} \frac{x^T R x}{\sqrt{x^T R x}}
\]

(13)

3 Classical concepts of correlation parameters and their pitfalls

3.1 Set-up of numerical examples

We will use some numerical examples to highlight the pitfalls of the classical concepts of correlation parameters when risks are not elliptically distributed. In order to avoid challenges arising from Monte Carlo simulations such as sampling error, our examples have been constructed such that the distribution functions are available in analytical form.

We assume that the losses \( X_i, i \in \{1, \ldots, n\} \), are independent and Gamma distributed with shape parameter \( \gamma_i \) and rate parameter \( \vartheta_i \). If all rate parameters are equal, \( \vartheta_1 = \ldots = \vartheta_n =: \vartheta \), the aggregate loss \( X \) is Gamma distributed with shape parameter \( \gamma_1 + \ldots + \gamma_n \) and rate parameter \( \vartheta \). In case of the rate parameters being not all the same, the analytical representation of the distribution function of \( X \) provided by Moschopoulos (1985, p. 543) is applied, which we denote by

\[
F_X(x) = F_{\Gamma+}(x; \gamma_1, \ldots, \gamma_n, \vartheta_1, \ldots, \vartheta_n)
\]

(14)
In the examples, we have calculated the true aggregate Value-at-Risk in (4) by inverting the distribution function based on the Newton method.

To determine the Value-at-Risk of a linear combination of risks in (5), note that for a scalar \( u_i > 0 \) the product \( u_i \cdot X_i \) is Gamma distributed with shape parameter \( \gamma_i \) and rate parameter \( \vartheta_i/u_i \). Hence, the distribution function of the linear combination \( u_1 X_1 + \ldots + u_n X_n \) is given by

\[
F_{\Gamma^+}(x; \gamma_1, \ldots, \gamma_n, \vartheta_1/u_1, \ldots, \vartheta_n/u_n)
\]

Partial derivatives, as required for the true marginal capital requirements in (6), have been calculated numerically. To obtain the distribution of \( X + h \cdot X_i \) for small values of \( h \), we have used the representation in (15) for a vector \( u \) with \( u_i = 1 + h \) and \( u_j = 1 \) for all \( j \neq i \).

We consider some calibrations of the Gamma distributed risks, which are designed as simple as possible to highlight the potential pitfalls. In all calibrations, the shape and rate parameter of each random variable \( X_i \) coincide, meaning that we always have \( \mathbb{E}[X_i] = \gamma_i/\vartheta_i = 1 \). We consider two alternative values for the parameters. Either we set \( \vartheta_i = \gamma_i = 0.5 \), which reflects a relatively heavy-tailed risk with a high coefficient of variation of 141.4% and a high ratio between the CapR_{99.5%} and the CapR_{90%} of 4.034,\(^6\) or we set \( \vartheta_i = \gamma_i = 2 \), which reflect a relatively light-tailed risk with a low coefficient of variation of 70.7% and a low ratio between the CapR_{99.5%} and the CapR_{90%} of 2.874.\(^7\)

\(^6\)For comparison, Bernard et al. (2018, p. 847) assume the distribution \( 200 \cdot \text{LogNormal}(0, 1) \) for aggregate Non-Life insurance risks. This implies a coefficient of variation of 131.1% and a ratio between the capital requirements at a 99.5% confidence level and a 90% confidence level of 5.884.

\(^7\)Bernard et al. (2018, p. 847) assume a normal distribution for the aggregate market risk, which implies a ratio between the 99.5% and 90% capital requirements of 2.01. This value is achieved by the Gamma distribution when setting the shape parameter \( k \) to infinity.
Finally, we distinguish between two types of risk calibrations: in risk calibration type I, all random variables $X_i$ have identical parameters. In risk calibration type II, we consider combinations of heavy-tailed and light-tailed risks, as defined by Table 1.

**Table 1: Risk calibrations of type II**

<table>
<thead>
<tr>
<th>Name of calibration</th>
<th>$n$</th>
<th>$\vartheta_1 = \gamma_1$</th>
<th>$\vartheta_2 = \gamma_2$</th>
<th>$\vartheta_3 = \gamma_3$</th>
<th>$\vartheta_4 = \gamma_4$</th>
<th>$\vartheta_5 = \gamma_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>IIa</td>
<td>2</td>
<td>0.5</td>
<td>2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>IIb</td>
<td>3</td>
<td>0.5</td>
<td>0.5</td>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>IIc</td>
<td>5</td>
<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
<td>2</td>
<td>2</td>
</tr>
</tbody>
</table>

Throughout section 3, in line with Solvency II, the capital requirement for risk $Y$ corresponds to the Value-at-Risk:

$$\text{CapR}_{1-\alpha}(Y) = \text{VaR}_{1-\alpha}(Y - \mathbb{E}(Y)) = \text{VaR}_{1-\alpha}(Y) - \mathbb{E}(Y)$$ (16)

and the confidence level is set to $1 - \alpha = 99.5\%$.

### 3.2 Pearson and pairwise VaR-implied correlations

This section identifies systematic biases of the square-root formula if it is calibrated with Pearson correlations or pairwise VaR-implied correlations. In terms of Pearson correlations, our analysis refers to the one by Pfeifer and Strassburger (2008), who demonstrate for independent Beta distributed risks that the square-root formula can substantially understate or overstate the aggregate capital requirement. We will demonstrate that pairwise VaR-implied correlations can induce even larger misstatements.
The Pearson correlation matrix is in our set-up the $n \times n$ identity matrix, due to the independence of the $X_i$. Pairwise VaR-implied correlations are defined based on pairs of random variables $X_i, X_j$, as

$$
\varrho_{i,j} = \frac{\text{CapR}_{1-\alpha}(X_i + X_j)^2 - \text{CapR}_{1-\alpha}(X_i)^2 - \text{CapR}_{1-\alpha}(X_j)^2}{2 \cdot \text{CapR}_{1-\alpha}(X_i) \cdot \text{CapR}_{1-\alpha}(X_j)} 
$$

(17)

if $i \neq j$ and otherwise $\varrho_{i,i} = 1$. For example, for two heavy-tailed risks with $\vartheta_i = \gamma_i = 0.5$, this results in

$$
\text{CapR}_{99.5\%}(X_1) = \text{VaR}_{99.5\%}(X_1) - \mathbb{E}(X_1) = 7.879 - 1 = 6.879
$$

$$
\text{CapR}_{99.5\%}(X_1 + X_2) = \text{VaR}_{99.5\%}(X_1 + X_2) - \mathbb{E}(X_1 + X_2) = 10.597 - 2 = 8.597
$$

$$
\varrho_{1,2} = \frac{8.597^2 - 6.879^2 - 6.879^2}{2 \cdot 6.879 \cdot 6.879} = -0.219
$$

(18)

For $n \geq 3$, all entries of the pairwise VaR-implied matrix $R$ aside from the diagonal are identical due to the symmetry of the risk calibrations of type I.

We measure the bias based on the relative error in line with Pfeifer and Strassburger (2008) as

$$
\text{Relative Error} = \frac{\sqrt{x^T R x}}{\text{CapR}_{1-\alpha}(X)} - 1
$$

(19)

Table 2 depicts the results for $n = 2, 4$ or 6 identically distributed risks (i.e. calibration type I). If $R$ is the Pearson correlation matrix, the square-root formula overstates the aggregate capital requirement by up to 34.3% in case of $n = 6$ heavy-tailed risks. If $R$ includes the pairwise VaR-implied correlations, the square-root formula determines the aggregate capital requirement in line with the true risk distribution as long as there are

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only \( n = 2 \) risks. If \( n \geq 3 \), the square-root formula may substantially understate the aggregate capital requirement (for instance, by \(-55.9\%\) if \( \vartheta = 2 \) and \( n = 6 \)). Moreover, \( x^T R x \) can be negative meaning that \( \sqrt{x^T R x} \) is not defined in real numbers. For instance, taking up on the example that \( \vartheta = 0.5 \) in connection with \( n = 6 \) risks,

\[
x^T R x = n \cdot \text{CapR}_{99.5\%}(X_1)^2 + n \cdot (n - 1) \cdot \varrho_{1,2} \cdot \text{CapR}_{99.5\%}(X_1)^2
\]

\[
= 6 \cdot 6.879^2 + 30 \cdot (-0.219) \cdot 6.879^2 = -27.308
\]

Table 2: Relative error of the square-root formula (\( R \) includes Pearson correlations or pairwise VaR-implied correlations) in terms of the aggregate capital requirement based on risk calibrations type I. “n/a” means that \( \sqrt{x^T R x} \) is not defined because \( x^T R x < 0 \).

<table>
<thead>
<tr>
<th>( \vartheta_i = \gamma_i ) (for all risks i)</th>
<th>Pearson</th>
<th>Pairwise VaR-implied</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n = 2 )</td>
<td>( n = 4 )</td>
<td>( n = 6 )</td>
</tr>
<tr>
<td>0.5 (heavy-tailed)</td>
<td>13.2%</td>
<td>26.7%</td>
</tr>
<tr>
<td>2 (light-tailed)</td>
<td>10.1%</td>
<td>18.9%</td>
</tr>
</tbody>
</table>

In addition to the aggregate capital requirement, the square-root formula can provide biased marginal capital requirements. To demonstrate this, we consider the risk calibrations of type II, i.e. the \( X_i \) do not have all the same parameters.\(^9\) The true marginal capital requirements according to (6) are \((6.6523, 0.4042)^T\) for calibration IIa, \((4.2072, 4.2072, 0.3334)^T\) for IIb, and \((3.1727, 3.1727, 3.1727, 0.2963, 0.2963)^T\) for IIc. Based on the square-root formula, the marginal capital requirements are determined by (12).

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\(^9\)If both risks were identically parameterized, the square-root formula as well as the true distribution imply that all risks receive the same marginal capital requirements due to symmetry. Hence, the relative error of marginal capital requirements would coincide with the ones of the aggregate capital requirement.
For example, for IIa, the marginal capital requirements based on the VaR-implied correlation are\(^{10}\)

\[
\frac{(Rx) \odot x}{\sqrt{x^T R x}} = \frac{1}{7.0565} \begin{pmatrix} 1 & -0.1313 \\ -0.1313 & 1 \end{pmatrix} \begin{pmatrix} 6.879 \\ 2.715 \end{pmatrix} \odot \begin{pmatrix} 6.879 \\ 2.715 \end{pmatrix} = \begin{pmatrix} 6.3593 \end{pmatrix}
\]

Hence, they are understated by \(6.3593/6.6523 - 1 = -4.4\%\) for risk 1 and overstated by \(0.6972/0.4042 - 1 = 72.5\%\) for risk 2. Note that for \(n = 2\), there is only one free parameter in the matrix \(R\) (given that it has ones on the diagonal). Given that the VaR-implied correlation parameter is the only one that makes the square-root formula consistent to the true aggregate capital requirement, there is no parameter in this example which makes the square-root formula consistent with the marginal capital requirements.

Table 3 shows the relative errors of the marginal capital requirements for risk calibrations IIa, b and c. It highlights that the square-root formula’s bias of marginal capital requirements can even be larger than the bias of the aggregate capital requirement.

Table 3: Relative error of the square-root formula (\(R\) includes Pearson correlations or pairwise VaR-implied correlations) in terms of the marginal capital requirement for each risk \(i\) based on the risk calibrations IIa-c.

<table>
<thead>
<tr>
<th>Calibr.</th>
<th>Pearson</th>
<th>Pairwise VaR-implied</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(i = 1)</td>
<td>(i = 2)</td>
</tr>
<tr>
<td>IIa</td>
<td>-3.8%</td>
<td>146.6%</td>
</tr>
<tr>
<td>IIb</td>
<td>11.4%</td>
<td>11.4%</td>
</tr>
<tr>
<td>IIc</td>
<td>19.2%</td>
<td>19.2%</td>
</tr>
</tbody>
</table>

\(^{10}\)The stand-alone capital requirements are \(x = (6.879, 2.715)^T\), the true aggregate capital requirement is 7.0565 and the VaR-implied correlation parameter is calculated analogously to (18) as \(-0.1313\).
3.3 Joint VaR-implied correlations

Mittnik (2014, p. 70 f.) proposes to determine the entries of $R$ jointly for a set of $\ell$ calibration portfolios which include portions of the risks that are to be aggregated. To this end, one identifies those correlation parameters $\varrho_{i,j}$ that minimize the expression

$$
\sum_{k=1}^{\ell} \left[ \text{CapR}_{1-\alpha}(\sum_{i=1}^{n} w_i^{(k)} X_i)^2 - \sum_{i=1}^{n} \sum_{j=1}^{n} \varrho_{ij} \text{CapR}_{1-\alpha}(w_i^{(k)} X_i)\text{CapR}_{1-\alpha}(w_j^{(k)} X_j) \right]^2 \tag{20}
$$

where $w_i^{(k)}$ is the weight of risk $i$ in portfolio $k$. For a given set of portfolios and assuming that $\varrho_{ij} = \varrho_{ji}$ for all $i, j \in \{1, \ldots, n\}$ and $\varrho_{ii} = 1$ for all $i \in \{1, \ldots, n\}$, Mittnik (2014) derives closed-form solutions for the set of correlation parameters $\varrho_{i,j}$ minimizing (20).

There are various possible sets of calibration portfolios. Mittnik (2014) distinguishes between an “exact” identification if the number of portfolios coincides with the number of correlation parameters to be determined, i.e. $\ell = n(n-1)/2$, and an “overidentified” identification if the number of portfolios exceeds that of the correlation parameters, i.e. $\ell > n(n-1)/2$. With an exact identification, the target function in (20) attains zero and hence, the square-root formula determines the capital requirement accurately for all $n(n-1)/2$ portfolios. As Proposition 1 states, there is always an exact identification of $R$ such that the square-root formula determines the aggregate capital requirement of a company in line with the true risk distribution.

**Proposition 1.** Let $x_i = \text{CapR}_{1-\alpha}(X_i) \neq 0$ for all $i \in \{1, \ldots, n\}$. Then there exist calibration portfolios such that $R$ defined by (20) fulfills $\sqrt{x^TRx} = \text{CapR}_{1-\alpha}(X)$.

In general for $n \geq 3$, the matrix $R$ that solves $\sqrt{x^TRx} = \text{CapR}_{1-\alpha}(X)$ is not unique and there is leeway in terms of the resulting marginal capital requirements. To demonstrate this, we choose in line with Mittnik (2014) equally-weighted calibration portfolios of up
to \( n \) risks. Table 4 sketches all possible equally-weighted portfolios for \( n = 5 \). There are \((\frac{5}{2}) = 10\) equally-weighted portfolios with 2 risks, \((\frac{5}{3}) = 10\) portfolios with 3 risks, \((\frac{5}{4}) = 5\) portfolios with 4 risks and one portfolio with 5 risks.

Table 4: Possible equally-weighted portfolios to calibrate \( R \) in case of \( n = 5 \) risks.

<table>
<thead>
<tr>
<th>Portfolio no., ( k )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>...</th>
<th>10</th>
<th>11</th>
<th>...</th>
<th>20</th>
<th>21</th>
<th>...</th>
<th>25</th>
<th>26</th>
</tr>
</thead>
<tbody>
<tr>
<td>( w_1 )</td>
<td>1/2</td>
<td>1/2</td>
<td>1/2</td>
<td>...</td>
<td>0</td>
<td>1/3</td>
<td>...</td>
<td>0</td>
<td>1/4</td>
<td>...</td>
<td>0</td>
<td>1/5</td>
</tr>
<tr>
<td>( w_2 )</td>
<td>1/2</td>
<td>0</td>
<td>0</td>
<td>...</td>
<td>0</td>
<td>1/3</td>
<td>...</td>
<td>0</td>
<td>1/4</td>
<td>...</td>
<td>1/4</td>
<td>1/5</td>
</tr>
<tr>
<td>( w_3 )</td>
<td>0</td>
<td>1/2</td>
<td>0</td>
<td>...</td>
<td>0</td>
<td>1/3</td>
<td>...</td>
<td>1/3</td>
<td>1/4</td>
<td>...</td>
<td>1/4</td>
<td>1/5</td>
</tr>
<tr>
<td>( w_4 )</td>
<td>0</td>
<td>0</td>
<td>1/2</td>
<td>...</td>
<td>1/2</td>
<td>0</td>
<td>...</td>
<td>1/3</td>
<td>1/4</td>
<td>...</td>
<td>1/4</td>
<td>1/5</td>
</tr>
<tr>
<td>( w_5 )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>...</td>
<td>1/2</td>
<td>0</td>
<td>...</td>
<td>1/3</td>
<td>0</td>
<td>...</td>
<td>1/4</td>
<td>1/5</td>
</tr>
</tbody>
</table>

For an exact identification, we need a set which comprises of \( 5 \cdot (5 - 1)/2 = 10 \) calibration portfolios. To ensure that the square-root formula determines the aggregate capital requirement in line with the true risk distribution, portfolio no. 26 needs to be within this set. The nine other portfolios can be arbitrarily chosen from Table 4, which gives us a choice among \((\frac{25}{9}) = 2,042,975\) sets of portfolios. Calibrating \( R \) based on all these sets would require extensive calculation time. For simplicity, we have performed the calibration for 1,000 randomly chosen sets of portfolios. Based on each of these sets, we have also determined the marginal capital requirements resulting from the square-root formula in connection with \( R \) according to (10). Figure 1 depicts for each set the marginal capital requirements of risks \( X_1 \) and \( X_4 \). All calibration portfolios lead to the true aggregate capital requirement, but in many cases the calculated marginal capital requirements deviate strongly from the results implied by the true risk distribution and thereby give the insurance company an odd view on the interplay of the risks. For example, several sets of portfolios imply that the marginal capital requirement of \( X_4 \) is negative, indicating that this risk provides a natural hedge of the aggregate risk and that an extension of the exposure to \( X_4 \) reduces the aggregate risk. In fact, however, the true aggregate capital requirement increases by such an extension. Moreover, in most cases the calculated
marginal capital requirements of risks 1, 2 and 3 are not identical; the same applies to risks 4 and 5. Hence, the insurance company would rank these risks differently, even though they are in fact identical.

Figure 1: Marginal capital requirement of risks $X_1$ and $X_4$ according to square-root formula in connection with matrix $R$ calibrated based on 1,000 randomly chosen sets of portfolios.

3.4 Correlations implied by error minimization

Finally, we investigate how closely the square-root formula (with ones on the diagonal of $R$) can approximate the first and second-order partial derivatives of the true aggregate capital requirement with respect to exposure changes. To this end, we identify the correlation parameters which minimize the error of the square-root formula in this sense. We measure this error by the mean squared differences between the entries of the gradients,

$$
\left\| \nabla_u g(u)_{|u=(1,\ldots,1)^T} - \nabla_u f(u)_{|u=(1,\ldots,1)^T} \right\|_2^2
$$

(21)
or by the mean squared differences between the entries of the Hesse matrices

$$\left\| H_u g(u) \big|_{u=(1,...,1)^T} - H_u f(u) \big|_{u=(1,...,1)^T} \right\|_2^2$$  \hspace{1cm} (22)

or by a weighted sum of (21) and (22). We consider two examples based on risk calibrations IIb and c. With calibration IIb, we have $n = 3$ risks and hence three parameters in $R$. As there are three entries in $\nabla_u f(u)$, we find a unique matrix $R$ which reduces the error in (21) to zero, namely

$$R = \begin{pmatrix}
1 & -0.175 & -0.119 \\
-0.175 & 1 & -0.119 \\
-0.119 & -0.119 & 1
\end{pmatrix}$$

However, based on these correlations, the square-root formula does not provide the correct Hesse matrix:

$$H_u g(u) = \begin{pmatrix}
3.3867 & -2.9717 & -0.415 \\
-2.9717 & 3.3867 & -0.415 \\
-0.415 & -0.415 & 0.830
\end{pmatrix} \neq H_u f(u) = \begin{pmatrix}
4.3408 & -4.1190 & -0.2222 \\
-4.1190 & 4.3408 & -0.2222 \\
-0.2222 & -0.2222 & 0.4444
\end{pmatrix}$$

Section 5.3 will demonstrate that a biased Hesse matrix can mislead an insurer’s risk-return optimization, as a saddle point can be mixed up with a local optimum.
Alternatively, it was possible to choose $R$ such that the error of the Hesse matrix in (22) is almost zero:

$$
R = \begin{pmatrix}
1 & -0.5088 & 0.2326 \\
-0.5088 & 1 & 0.2326 \\
0.2326 & 0.2326 & 1
\end{pmatrix}
$$

Based on these correlations, the square-root formula provides biased marginal capital requirements:

$$
\nabla u g(u) = \begin{pmatrix}
3.2689 \\
3.2689 \\
1.9029
\end{pmatrix} \neq \nabla u f(u) = \begin{pmatrix}
4.2072 \\
4.2072 \\
0.3334
\end{pmatrix}
$$

Consequently, it was not possible to choose $R$ such that the (weighted) sum of (21) and (22) was zero.

4 Partial-derivative implied correlations

4.1 Approach

To eliminate the weaknesses of the square-root formula that were identified in section 3, we propose an alternative view on the meaning of the matrix $R$. If the random vector $(X_1, \ldots, X_n)$ follows an elliptical distribution and the risk measure $\varrho$ is as defined in section 2, the capital requirement based on the true risk distribution can be stated as (cf. McNeil et al., 2015, pp. 295)

$$
f_X(u) = \text{CapR}\left(\sum_{i=1}^{n} u_i \cdot X_i\right) = \varrho\left(\sum_{i=1}^{n} u_i \cdot X_i\right) - \mathbb{E}\left(\sum_{i=1}^{n} u_i \cdot X_i\right) = \sqrt{\sum_{i,j=1}^{n} \varrho_{ij} u_i x_i u_j x_j},
$$
where \( x_i \) denote the stand-alone capital requirements of \( X_i \) (cf. (2)). This implies

\[
\frac{\partial}{\partial u_k} f_X^2(u) = 2 x_k \sum_{j=1}^{n} \rho_{kj} u_j x_j, \tag{23}
\]

and

\[
\rho_{k\ell} = \frac{1}{2 x_k x_\ell} \frac{\partial^2}{\partial u_k \partial u_\ell} f_X^2(u) \tag{24}
\]

Hence, the entries of \( R \) coincide with the second-order partial derivative of the squared capital requirement with respect to exposures to the respective risks, divided by twice the stand-alone capital requirements of these risks. For elliptical distributions, Eq. (24) holds globally for any exposure vector \( u \).

If the risks are not elliptically distributed, Eq. (24) does not hold globally. Nevertheless, Eq. (24) provides a unique definition of a matrix \( R = (\rho_{k\ell})_{k,\ell=1}^{n} \). Proposition 2 shows that the square-root formula in connection with this matrix provides a proper local approximation of the true aggregate capital requirement at the actual portfolio, \( u = 1_n \).

To derive Proposition 2, we first formulate Theorem 1, which states that any homogeneous and twice continuously differentiable function can be locally approximated by a function which corresponds to the square-root formula.

**Theorem 1.** Let \( U \subseteq \mathbb{R}^n \) be open, \( x \in U \), and the function \( f : U \to \mathbb{R} \) be homogeneous of degree one and twice continuously differentiable. Let \( f(x) > 0 \). Then, the matrix \( R = R(x) = (\rho_{k\ell})_{k,\ell=1}^{n} \) defined by

\[
\rho_{k\ell} = \rho_{k\ell}(x) = \frac{1}{2} \frac{\partial^2}{\partial x_k \partial x_\ell} f^2(x) \tag{25}
\]
is symmetric. Writing

\[ f_x(u) = f(u \circ x) \quad (26) \]

and

\[ g_x(u) = \sqrt{(u \circ x)^T R(x)(u \circ x)} \quad (27) \]

the following holds:

\[ g_x(1_n) = f_x(1_n), \quad (28) \]

\[ \frac{\partial}{\partial u_\ell} g_x(1_n) = \frac{\partial}{\partial u_\ell} f_x(1_n), \quad 1 \leq \ell \leq n, \quad (29) \]

\[ \frac{\partial^2}{\partial u_k \partial u_\ell} g_x(1_n) = \frac{\partial^2}{\partial u_k \partial u_\ell} f_x(1_n), \quad 1 \leq k, \ell \leq n. \quad (30) \]

Note that the function \( g_x(u) \) in (27) can be a more useful approximation of the original function \( f_x(u) \) in (26) than a Taylor polynomial, since \( g_x(u) \) is like \( f_x(u) \) homogeneous of degree one in \( u \);\(^{11}\) a Taylor polynomial of degree two (or higher) does not have this property and a Taylor polynomial of degree one could not approximate the original function up to second-order partial derivatives. Proposition 2 translates Theorem 1 to our context.

**Proposition 2.** Let \( \varrho \) be a homogeneous, translation-invariant and law-invariant risk measure and let \((X_1, \ldots, X_n)\) be a risk vector. Let \( M \subseteq \mathbb{R}^n \) be open with \( 1_n \in M \). Assume that the function

\[ u = (u_1, \ldots, u_n)^T \mapsto f_X(u) = \varrho \left( \sum_{i=1}^n u_i \cdot X_i \right) - \mathbb{E} \left( \sum_{i=1}^n u_i \cdot X_i \right) \quad (31) \]

\(^{11}\)In our context, this assumption is needed for the approximate risk aggregation formula to induce a homogeneous risk measure and for the Euler capital allocation to add up to the aggregate capital requirement.
is twice continuously differentiable on $M$ with $f_X(1_n) > 0$. The entries of the vector $x \in \mathbb{R}^n$ are defined as

$$x_i = \varrho(X_i) - \mathbb{E}(X_i)$$  \hfill (32)

Then, the matrix $R = (\varrho_{k\ell})_{k,\ell=1}^{n}$ defined by

$$\varrho_{k\ell} = \frac{1}{2x_kx_\ell} \frac{\partial^2}{\partial u_k \partial u_\ell} f_X^2(1_n)$$  \hfill (33)

is symmetric. Writing $g_x(u)$ as

$$g_x(u) = \sqrt{(u \circ x)^T R (u \circ x)},$$  \hfill (34)

the following holds:

$$g_x(1_n) = f_X(1_n),$$  \hfill (35)

$$\frac{\partial}{\partial u_\ell} g_x(1_n) = \frac{\partial}{\partial u_\ell} f_X(1_n), \quad 1 \leq \ell \leq n,$$  \hfill (36)

$$\frac{\partial^2}{\partial u_k \partial u_\ell} g_x(1_n) = \frac{\partial^2}{\partial u_k \partial u_\ell} f_X(1_n), \quad 1 \leq k, \ell \leq n.$$  \hfill (37)

We call the matrix whose entries are defined in (33) the “partial-derivative implied tail correlation matrix”.\footnote{Similarly to the term “VaR-implied tail correlation matrix”, which has been used for example by Mittnik (2014), we call the matrix a tail correlation matrix in order to point out that it does not have the properties of an ordinary correlation matrix. With regard to the entries of the matrix, we will talk of partial-derivative implied correlations, i.e. we will omit the “tail” for simplicity.} In connection with this matrix, important properties of the square-root formula are generalized from elliptical distributions towards almost arbitrary multivariate distributions (as long as the capital requirement is twice continuously differentiable): for the portfolio based on which the correlations are calibrated, the square-root
formula provides the aggregate capital requirement, the marginal capital requirements of all risks (cf. Eq. (12)) as well as all second-order derivatives of the aggregate capital requirement (cf. Eq. (13)) in accordance with the true multivariate risk distribution. In the special case of elliptical distributions, the partial-derivative implied tail correlation matrix coincides with the Pearson correlation matrix and hence, its diagonal elements are one. For general distributions, the diagonal elements may deviate from one. For instance, for risk calibration IIa, we have

\[ R = \begin{pmatrix}
1.0244 & -0.0824 \\
-0.0824 & 0.5958
\end{pmatrix} \] (38)

Appendix D provides the calculations of the first and second-order partial derivatives of the aggregate capital requirement based on the square-root formula in connection with this matrix, which are all in line with results based on the true risk distribution.

### 4.2 Numerical computation

The computation of the partial-derivative implied correlations requires calculating the second-order partial derivatives of the Value-at-Risk based on the true multivariate distribution with respect to exposure changes. As explained in section 3.1, these calculations can be easily done numerically if the distribution function of linear combinations of the \( X_i \) has an analytical representation. The set-up of independent Gamma distributed random variables, as assumed in section 3.1, is quite restrictive for practical considerations. Furman et al. (2019) propose the class of multivariate mixed-gamma distributions, which is flexible in terms of the shape of the univariate distributions and the stochastic dependencies between them. In fact, it may approximate any continuous multivariate distribution
with non-negative supports arbitrarily well.\footnote{Furman et al. (2019) show in Theorem 4 that the class of mixed-gamma distributions is dense in the class of continuous multivariate distributions with non-negative supports. More precisely, for any random vector in the latter class, a sequence of mixed-gamma distributed random vectors can be constructed which converges in distribution to the given random vector.} Also, as we explain in the next paragraph, this class allows for an analytical representation of the distribution function of linear combinations of the univariate random variables.

According to Furman et al. (2019, p. 8 f.), the $n$-dimensional mixed-gamma distribution is defined as follows. Let $\kappa = (\kappa_1, \ldots, \kappa_n)$ be a vector of discrete random variables which can assume non-negative integer values and let $p_\kappa(k) = \mathbb{P}(\kappa_1 = k_1, \ldots, \kappa_n = k_n)$ denote the probability mass function of $\kappa$ with $k = (k_1, \ldots, k_n) \in \mathbb{N}^n_0$. Let $f_\Gamma(x; \gamma, \vartheta)$ denote the density function of the univariate Gamma distribution with shape parameter $\gamma$ and rate parameter $\vartheta$. The random vector $\Gamma^{(\kappa)} = (\Gamma_1^{(\kappa_1)}, \ldots, \Gamma_n^{(\kappa_n)})$ is distributed $n$-variate mixed-gamma if its density function is given by

$$f_{\Gamma^{(\kappa)}}(x_1, \ldots, x_n) = \sum_{k \in \mathbb{N}^n_0} p_\kappa(k) \prod_{i=1}^n f_\Gamma(x_i; \gamma_{k_i}, \vartheta_i)$$  \hspace{1cm} (39)

where the shape parameters are determined by $\gamma_{k_i} = \gamma_i + k_i$ with $\gamma_i > 0$. Recall that $F_{\Gamma^{+}}(x; \gamma_1, \ldots, \gamma_n, \vartheta_1, \ldots, \vartheta_n)$ in (14) is the distribution function of the sum of independent Gamma distributed random variables with different shape and rate parameters. Assum-
ing that there is only a finite number of vectors $k$ with positive probability $p_\kappa(k)$, the
distribution function of $X = \Gamma_1^{(\kappa_1)} + \ldots + \Gamma_n^{(\kappa_n)}$ is given by

$$F_X(x) = \int \ldots \int \sum_{k \in \mathbb{N}_0^n} p_\kappa(k) \prod_{i=1}^n f_\Gamma(y_i; \gamma_{ki}, \vartheta_i) dy_1 \ldots dy_n$$

$$= \sum_{k \in \mathbb{N}_0^n} p_\kappa(k) \int \ldots \int \prod_{i=1}^n f_\Gamma(y_i; \gamma_{ki}, \vartheta_i) dy_1 \ldots dy_n$$

$$= \sum_{k \in \mathbb{N}_0^n} p_\kappa(k) F^{\Gamma_+}(x; \gamma_{k_1}, \ldots, \gamma_{k_n}, \vartheta_1, \ldots, \vartheta_n),$$

and the distribution function of the linear combination $u_1 \Gamma_1^{(\kappa_1)} + \ldots + u_n \Gamma_n^{(\kappa_n)}$ is given by

$$\sum_{k \in \mathbb{N}_0^n} p_\kappa(k) F^{\Gamma_+}(x; \gamma_{k_1}, \ldots, \gamma_{k_n}, \vartheta_1/u_1, \ldots, \vartheta_n/u_n)$$

5 Implications for business steering

5.1 Set-up

This section illustrates that a proper specification of $R$ is highly relevant for business
steering. Biases in terms of marginal capital requirements or the Hesse matrix can have
detrimental consequences for policyholders in terms of insurance premiums as well as the
safety level that the insurer actually attains. Moreover, we demonstrate that partial-
derivative implied correlations provide appropriate steering signals, even if the insurer’s
optimal portfolio is relatively far away from the portfolio based on which the correlations
have been calibrated.

Assume that an insurer operates with $n$ lines of business (lob’s). The scalars $u_i$ represent
the volume of lob $i$ in terms of the number of insurance contracts. Suppose that the
$u_i$ are scaled, for example, in 100,000 contracts such that we may disregard the integer restriction. Moreover, we assume that the diversification within each lob does not vary in $u_i$ such that the total claims costs of lob $i$ are modelled by $u_i \cdot X_i$. The connection between the volume $u_i$ and the premium $p_i$ of lob $i$ is determined by an isoelastic demand function,

$$u_i(p_i) = n_i \cdot p_i^{\epsilon_i},$$

(40)

where $n_i > 0$ calibrates demand to market size and $\epsilon_i < -1$ is the price elasticity of demand which is constant in $p_i$. We consider a representative insurer whose objective is to maximize its economic value added (EVA).\(^{15}\) In our model, the insurer’s EVA is the expected profit, deducted by the cost of capital, which are modelled by a hurdle rate $r_h$ times the insurer’s capital requirement:

$$\text{EVA}(u) = \sum_{i=1}^{n} u_i \cdot (p_i(u_i) - \mathbb{E}[X_i]) - r_h \cdot \text{CapR}_{1-\alpha} \left( \sum_{i=1}^{n} u_i \cdot X_i \right),$$

where $p_i(u_i)$ is the inverse of the demand function in Eq. (40).

The EVA-maximizing strategy can be identified by the Newton method. To this end, let $\nabla_u \text{EVA}$ denote the gradient of the EVA with respect to $u$ and $H_u \text{EVA}$ the respective Hessian matrix. Starting from the vector $u^{(k)} \in \mathbb{R}^n$, the next iteration is to choose the volumes

$$u^{(k+1)} = u^{(k)} - [H_u \text{EVA}]^{-1} \nabla_u \text{EVA}$$

(41)

\(^{14}\)To simplify the notation, $p_i$ is also scaled. If $u_i$ are specified per 100,000 contracts, $p_i$ is the premium income per 100,000 contracts.

\(^{15}\)In the context of an insurer’s asset management, Braun et al. (2018) use the return on risk-adjusted capital (RORAC) as the insurer’s objective. In the context of evaluating the performance of an insurer’s lob’s, the EVA is advantageous, since the marginal capital of a lob can become negative (or zero) implying that the RORAC is not meaningful (or not defined); cf. Diers (2011, pp. 117-123), who derives those results in a realistically calibrated simulation study.

26
The optimal strategy satisfies $\nabla_u \text{EVA} = 0$, which can equivalently be expressed in terms of the optimal premiums

$$p_i = \frac{1}{1 + 1/\epsilon_i} \cdot \left( \mathbb{E}[X_i] + r_h \cdot \frac{\partial}{\partial u_i} \text{CapR}_{1-a} \left( \sum_{i=1}^{n} u_i \cdot X_i \right) \right) \quad \forall i \in \{1, ..., n\} \quad (42)$$

Hence, the optimal premium for lob $i$ reflects the expected claims costs within this lob, $\mathbb{E}[X_i]$, plus the additional cost of capital caused by a marginal extension of this lob, multiplied with a loading factor which is the larger, the closer the price elasticity of demand is at $-1$. In our numerical calculations, we will set $\epsilon_i = -9$ for all lob’s $i$, and $r_h = 5\%$.\(^{16}\)

### 5.2 Relevance of marginal capital requirements

This section quantifies the potential missteering effects of the square-root formula due to biased marginal capital requirements. We start with risk calibration IIa. As a benchmark case, we determine the Value-at-Risk based on the true risk distribution and derive the corresponding EVA-maximizing strategy. If the demand function parameters are $n_1 = 38.2576$ and $n_2 = 3.4561$, the EVA-maximizing strategy is characterized by $u = (1, 1)^T$. Hence, the true marginal capital requirements are those calculated in section 3.2 and the optimal premiums are determined by Eq. (42) as

$$p_1 = \frac{1}{1 + 1/(-9)} \cdot (1 + 5\% \cdot 6.6523) = 1.499$$

$$p_2 = \frac{1}{1 + 1/(-9)} \cdot (1 + 5\% \cdot 0.4042) = 1.148$$

\(^{16}\)According to the empirical results of Yow and Sherris (2008, p. 318), this may reflect the price elasticity of compulsory third party or motor insurance.

\(^{17}\)Zanjani (2002, p. 297) estimates that the discounted cost of holding capital are 5\% in commercial automobile insurance.
The larger marginal capital requirement of lob 1 with heavy-tailed risks makes insurance contracts substantially more expensive than those of lob 2. Figure 2 depicts the contour lines of the EVA-function. Point A, i.e. \( u = (1, 1)^T \), reflects the optimal strategy that has just been calculated. In addition, Table 5 shows all relevant calculation results.

Suppose now that the capital requirement in the EVA is determined by the square-root formula in connection with the partial-derivative implied tail correlation matrix, which is calibrated based on the exposures \( u = (1, 1)^T \). Given that the marginal capital requirements coincide with the true ones according to Theorem 1, the strategy \( u = (1, 1)^T \) and \( p = (1.499, 1.148)^T \) is again EVA-optimal.

Next, suppose that the square-root formula is used in connection with the VaR-implied matrix, which, again, is calibrated for \( u = (1, 1)^T \). Since the square-root formula understates the heavy-tailed risk (cf. Table 3), the insurer offers this risk at a reduced price and thereby increase the exposure to this risk. Vice versa, the insurer offers the light-tailed risk at an increased premium. In total, the optimal strategy is led astray to \( u = (1.0616, 0.9286)^T \) and \( p = (1.489, 1.1572)^T \). Recall that the VaR-implied matrix has been calibrated such that the square-root formula determines the aggregate capital requirement in accordance with the true risk distribution only for \( u = (1, 1)^T \). For the new portfolio, the square-root formula (slightly) understates the aggregate risk. If the insurer adjusts its equity capital in accordance with the capital required by the standard formula, the true VaR confidence level reduces to \( 1 - 0.509\% = 99.491\% \). In connection with Pearson correlations, the square-root formula again understates the heavy-tailed risk and overstates the light-tailed risk. Hence, the insurer again takes a larger exposure to heavy-tailed risk than based on the true risk distribution. However, since Pearson cor-
relations lead to an overstatement of the aggregate risk (cf. Table 2), the insurer holds
enough capital to increase the true VaR confidence level to \(1 - 0.44\% = 99.56\%\).

So far, we have made the strong assumption that the calibration portfolio for \(R\) coincides
with the EVA-maximizing portfolio based on the true risk distribution, whereas in practical considerations, these two portfolios might deviate.\(^{18}\) We will now demonstrate that those deviations hardly impact the EVA-maximizing strategy if partial-derivative implied
correlations are used, but they intensify the distortions caused by VaR-implied correlations.\(^{19}\) Suppose that the regulator calibrates \(R\) based on less heavy-tailed and more
light-tailed risks (point B in Figure 2).\(^{20}\) An insurer using the square-root formula opts
for \(u = (1.067, 0.947)^T\) if VaR-implied correlations are used and for \(u = (0.995, 1.006)^T\)
if partial-derivative implied correlations are used. The arrows in Figure 2 depict these
distortions graphically. The true VaR confidence level induced by the square-root formula
is \(1 - 0.524\%\) in case of VaR-implied correlations, but still at 99.5\% in case of partial-
derivative implied correlations. If the regulatory calibration portfolio is even more biased
towards light-tailed risks (point B’ in Figure 2), VaR-implied correlations distort the
ture VaR confidence level to \(1 - 0.54\%\), and partial-derivative implied correlations to
\(1 - 0.496\%\). For completeness, Table 5 shows the results for two additional calibration
portfolios C and D which are biased in other directions. The results for calibration port-
folio D are identical to those for A, since the exposures are scaled proportionally, which
leaves the correlation parameters unaffected.

\(^{18}\)For example, the regulatory standard formula shall fit to a large number of companies, which do not
have identical portfolios.

\(^{19}\)Note that Pearson correlations are unaffected by the calibration portfolio and hence there is no need
to discuss the results for Pearson correlations in this context.

\(^{20}\)The calibration portfolio \(u = (0.9, 1.1)^T\) means that the regulator assumes the true risk distribution
to be \((0.9X_1, 1.1X_2)^T\).
Table 5: EVA-optimal strategies and true VaR confidence levels based on square-root formula with Pearson, VaR-implied or partial-derivative implied (pd-impl.) correlations, calibrated at the coordinates A, B, B’, C and D (cf. Fig. 2); risk calibration IIa.

<table>
<thead>
<tr>
<th>Type of R</th>
<th>R calibrated at</th>
<th>Optimal volumes</th>
<th>Optimal premiums</th>
<th>1-True Conf. Level</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$u_1$ $u_2$</td>
<td>$u_1$ $u_2$</td>
<td>$p_1$ $p_2$</td>
<td></td>
</tr>
<tr>
<td>A Pearson</td>
<td>1.0 1.0</td>
<td>1.033 0.833</td>
<td>1.494 1.171</td>
<td>0.440%</td>
</tr>
<tr>
<td>VaR-impl.</td>
<td>1.0 1.0</td>
<td>1.062 0.929</td>
<td>1.489 1.157</td>
<td>0.509%</td>
</tr>
<tr>
<td>pd-impl.</td>
<td>1.0 1.0</td>
<td>1.000 1.000</td>
<td>1.499 1.148</td>
<td>0.500%</td>
</tr>
<tr>
<td>B VaR-impl.</td>
<td>0.9 1.1</td>
<td>1.067 0.947</td>
<td>1.488 1.155</td>
<td>0.524%</td>
</tr>
<tr>
<td>pd-impl.</td>
<td>0.9 1.1</td>
<td>0.995 1.006</td>
<td>1.500 1.147</td>
<td>0.500%</td>
</tr>
<tr>
<td>B’ VaR-impl.</td>
<td>0.8 1.2</td>
<td>1.073 0.965</td>
<td>1.488 1.152</td>
<td>0.540%</td>
</tr>
<tr>
<td>pd-impl.</td>
<td>0.8 1.2</td>
<td>0.975 1.026</td>
<td>1.504 1.144</td>
<td>0.496%</td>
</tr>
<tr>
<td>C VaR-impl.</td>
<td>1.1 0.9</td>
<td>1.057 0.912</td>
<td>1.490 1.160</td>
<td>0.497%</td>
</tr>
<tr>
<td>pd-impl.</td>
<td>1.1 0.9</td>
<td>0.998 1.003</td>
<td>1.500 1.147</td>
<td>0.500%</td>
</tr>
<tr>
<td>D VaR-impl.</td>
<td>0.9 0.9</td>
<td>1.062 0.929</td>
<td>1.489 1.157</td>
<td>0.509%</td>
</tr>
<tr>
<td>pd-impl.</td>
<td>0.9 0.9</td>
<td>1.000 1.000</td>
<td>1.499 1.148</td>
<td>0.500%</td>
</tr>
</tbody>
</table>

Figure 3 shows for a wide range of calibration portfolios that VaR-implied correlations distort the optimal portfolio in terms of its true VaR confidence level much more than partial-derivative implied correlations do. The coordinates in Figure 3 depict the calibration portfolio $\tilde{u} \in [0.7, 1.3] \times [0.7, 1.3]$. The colors in the graph on the left-hand side of Figure 3 depict the true confidence level if VaR-implied correlations are used. Here, only a narrow area of calibration portfolios ensures a true confidence level of 99.5%. According to the graph on the right-hand side, the partial-derivative implied tail correlation matrix ensures that the EVA-optimal strategy has a confidence level close to 99.5% for a wide area of calibration portfolios.
Figure 2: Contour lines of the function $EVA(u)$ based on true risk distribution. Points A, B, B', C and D reflect the portfolios based on which $R$ is calibrated. The arrows show how the portfolio is adjusted if the calibration portfolio is B and VaR-implied correlations (upper arrow) or partial-derivative implied correlations (“pd-implied”, lower arrow) are used.

Figure 3: True VaR confidence level of the EVA-optimal strategy based on square-root formula depending on the volumes $(\tilde{u}_1, \tilde{u}_2)$ based on which $R$ is calibrated.
We now study the EVA-optimal strategy for 3 risks (calibration IIb) and 5 risks (IIc). Again, we set the market size parameters $n_i$ of the demand function such that $u = 1_n$ is EVA-optimal if the capital requirement is based on the true risk distribution. The results are depicted in Tables 6 (IIb) and 7 (IIc). In all cases, the square-root formula in connection with partial-derivative implied correlations induces a strategy which is very close at the optimal one and corresponds to a true VaR confidence level of almost exactly 99.5%. For Pearson and pairwise\(^{21}\) VaR-implied correlations, the biases shown in Table 5 for $n = 2$ risks intensify with a larger number of risks. In case of Pearson correlations, the aggregate Value-at-Risk is increasingly overstated with a larger number of risks (cf. Table 2) and hence the true confidence level of the chosen portfolio increases further with a larger number of risks; for $n = 5$ risks it is $1 - 0.159\%$. In case of pairwise VaR-implied correlations, the aggregate Value-at-Risk is increasingly understated with a larger number of risks. Even if the calibration portfolio coincides with the optimal portfolio (A), the true confidence level of the chosen portfolio deteriorates to $1 - 1.398\%$ for $n = 5$ risks. If the calibration portfolio underweights heavy-tailed risks (B),\(^{22}\) the true confidence level reduces to $1 - 1.683\%$ for $n = 5$ risks. Next, we calibrate the square-root formula with correlations implied by minimizing the errors in (21) and (22). For $n = 3$ risks, the three correlation parameters in $R$ are uniquely determined by forcing the error of the gradient in (21) to be zero; for $n = 5$ risks, the ten correlation parameters are chosen such that the error of the Hesse matrix in (22) is minimal based on the condition that the error of the gradient is zero. Since the square-root formula accurately reflects the gradient, it induces the true optimal portfolio, i.e. $u = 1_n$, if the calibration portfolio is $1_n$ (A). If\(^{21}\) The implications of alternative specifications of VaR-implied correlations are discussed at the end of this subsection.\(^{22}\) For $n = 3$ risks, the calibration portfolio is $u_1 = u_2 = 0.9$ and $u_3 = 1.1$. For $n = 5$ risks, the calibration portfolio is $u_1 = u_2 = u_3 = 0.9$ and $u_4 = u_5 = 1.1$. 

\[^{21}\] The implications of alternative specifications of VaR-implied correlations are discussed at the end of this subsection.

\[^{22}\] For $n = 3$ risks, the calibration portfolio is $u_1 = u_2 = 0.9$ and $u_3 = 1.1$. For $n = 5$ risks, the calibration portfolio is $u_1 = u_2 = u_3 = 0.9$ and $u_4 = u_5 = 1.1$. 

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Table 6: EVA-optimal strategies and true VaR confidence levels based on square-root formula with Pearson correlations, pairwise VaR-implied (Pw. VaR-impl.) correlations, correlations minimizing the error in (21) or partial-derivative implied (pd-impl.) correlations, calibrated at the coordinates A, B, C; risk calibration IIb ($n = 3$ risks).

<table>
<thead>
<tr>
<th>Type of $R$</th>
<th>$R$ calibrated at</th>
<th>Optimal volumes</th>
<th>1-True Conf. Level</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$u_1 = u_2$</td>
<td>$u_3$</td>
<td>$u_1 = u_2$</td>
</tr>
<tr>
<td>A Pearson</td>
<td>1</td>
<td>1</td>
<td>0.839</td>
</tr>
<tr>
<td></td>
<td>Pw. VaR-impl.</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>Min. error</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>pd-impl.</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>B Pw. VaR-impl.</td>
<td>0.9</td>
<td>1.1</td>
<td>1.05</td>
</tr>
<tr>
<td></td>
<td>Min. error</td>
<td>0.9</td>
<td>1.1</td>
</tr>
<tr>
<td></td>
<td>pd-impl.</td>
<td>0.9</td>
<td>1.1</td>
</tr>
<tr>
<td>C Pw. VaR-impl.</td>
<td>1.1</td>
<td>0.9</td>
<td>1.04</td>
</tr>
<tr>
<td></td>
<td>Min. error</td>
<td>1.1</td>
<td>0.9</td>
</tr>
<tr>
<td></td>
<td>pd-impl.</td>
<td>1.1</td>
<td>0.9</td>
</tr>
</tbody>
</table>

the calibration portfolio is different, the gradients are only accurate at the coordinates of the calibration portfolio and the square-root formula induces a portfolio which differs from $I_n$ and has a true VaR confidence level of $1 - 0.32\%$ ($B, n = 5$ risks) or $1 - 0.734\%$ ($C, n = 5$ risks).

As an alternative to pairwise VaR-implied correlations, we have determined the EVA-optimal strategy based on the correlations resulting from the 1,000 sets of portfolios considered in section 3.3, which make the square-root formula consistent with the true aggregate Value-at-Risk. In each of the 1,000 instances, the EVA-optimal strategy results in a true confidence level below $99.5\%$. For 10% of the sets of calibration portfolios, the true confidence level is below $1 - 0.75\%$; in one case it is even $1 - 14.9\%$. Here, the EVA-optimal exposures to the heavy-tailed risks rank between 0.587 and 0.716 and those to the light-tailed risks are 2.359 and 1.988. The square-root formula understates the Value-at-Risk for these exposures which leads to a substantially lower confidence level than the originally desired one.
Table 7: EVA-optimal strategies and true VaR confidence levels based on square-root formula with Pearson correlations, pairwise VaR-implied (Pw. VaR-impl.) correlations, correlations minimizing the error in (21) or partial-derivative implied (pd-impl.) correlations, calibrated at the coordinates A, B, C; risk calibration IIc ($n = 5$ risks).

<table>
<thead>
<tr>
<th>Type of $R$</th>
<th>$R$ calibrated at</th>
<th>Optimal volumes</th>
<th>1-True Conf. Level</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$u_1 = u_2 = u_3$</td>
<td>$u_4 = u_5$</td>
<td></td>
</tr>
<tr>
<td>A Pearson</td>
<td>1</td>
<td>0.801</td>
<td>0.159%</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>1.19</td>
<td>1.398%</td>
</tr>
<tr>
<td>Min. error</td>
<td>1</td>
<td>1</td>
<td>0.5%</td>
</tr>
<tr>
<td>pd-impl.</td>
<td>1</td>
<td>1</td>
<td>0.5%</td>
</tr>
<tr>
<td></td>
<td>0.9</td>
<td>1.2</td>
<td>1.683%</td>
</tr>
<tr>
<td>Min. error</td>
<td>0.9</td>
<td>0.892</td>
<td>0.32%</td>
</tr>
<tr>
<td>pd-impl.</td>
<td>0.9</td>
<td>0.998</td>
<td>0.5%</td>
</tr>
<tr>
<td>B Pw. VaR-impl.</td>
<td>1.1</td>
<td>1.18</td>
<td>1.204%</td>
</tr>
<tr>
<td>Min. error</td>
<td>1.1</td>
<td>1.117</td>
<td>0.734%</td>
</tr>
<tr>
<td>pd-impl.</td>
<td>1.1</td>
<td>0.999</td>
<td>0.5%</td>
</tr>
<tr>
<td>C Pw. VaR-impl.</td>
<td>1.1</td>
<td>0.999</td>
<td>1.001</td>
</tr>
<tr>
<td>Min. error</td>
<td>1.1</td>
<td>1.001</td>
<td>0.5%</td>
</tr>
</tbody>
</table>

5.3 Relevance of the Hesse matrix

This section highlights that the square-root formula should accurately reflect the second-order partial derivatives of the aggregate capital requirement with respect to exposure changes, or, in other words, the Hesse matrix of the function $f_X(u)$. We make an example based on the mixed-gamma distribution from section 4.2 with the parameters defined in Table 8. With a large weight in terms of $p_\kappa$, the distribution is identical to a calibration used in section 3.2 with $n = 3$ independent and identically distributed heavy-tailed risks. However, conditioning on a high aggregate loss, the risks $X_1$ and $X_2$ are negatively correlated. In this set-up, the marginal capital requirement of $X_1$ decreases when increasing the exposure to $X_1$. Hence, the Hesse matrix has negative entries on the diagonal:

$$H_u f(u) = \begin{pmatrix} -3.850 & 3.632 & 0.218 \\ 3.632 & -3.850 & 0.218 \\ 0.218 & 0.218 & -0.437 \end{pmatrix}$$
The aggregate Value-at-Risk can be reduced by shifting the exposures from $u = (1, 1, 1)^T$ to $u = (1+h, 1-h, 1)^T$ for a small value of $h$. We embed this distribution into the EVA-optimization problem as studied in section 5.2. By setting $n_1 = n_2 = 128.082$ and $n_3 = 90.209$, the first-order derivatives of the function $EVA(u)$, with the capital requirement being calculated based on the true risk distribution, are zero for $u = (1, 1, 1)^T$. The Hesse matrix of $EVA(u)$ is indefinite at $u = (1, 1, 1)^T$ reflecting that it is a saddle point, as illustrated on the left side of Figure 4. To keep the example graphically fully tractable, we assume from now on that $u_3 = 1$ is fixed and only $u_1$ and $u_2$ are decision variables. Then the globally optimal portfolio based on the true risk distribution is given by $u_1 = 1.8365$ and $u_2 = 0.5998$ (due to symmetry, these values can be exchanged; cf. points B and B’ in Figure 4).

We calibrate the correlation parameters based on error minimization of (21) at the calibration portfolio $u = (1, 1, 1)^T$, which gives a unique result for $n = 3$ risks. Since the square-root formula reflects the marginal capital requirements accurately at $u = (1, 1, 1)^T$, the first-order derivatives of the EVA function in connection with the square-root formula are—correctly—zero at $u = (1, 1, 1)^T$. The Hesse matrix of $EVA(u)$ is negative definite, and the EVA function has a global maximum at $u = (1, 1, 1)^T$, as shown on the right side of Figure 4. The lower part of Figure 4 illustrates the EVA function based on the square-root formula with partial-derivative based correlations calibrated at $u = (1, 1, 1)^T$. At this point, the VaR function (and hence the EVA function) adequately approximate the corresponding functions based on the true risk distribution, and hence, the company does not mistake the saddle point as an optimum.

\textsuperscript{23}This can be seen by approximating $f(u)$ by a Taylor polynomial of degree 2 and noting that $\partial f(u)/\partial u_1 = \partial f(u)/\partial u_2$ at $u = (1, 1, 1)^T$. 

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Table 8: Parameters of the mixed-gamma distribution.

<table>
<thead>
<tr>
<th>$i$</th>
<th>$\gamma_1$</th>
<th>$\gamma_{k_1}$</th>
<th>$\gamma_{k_2}$</th>
<th>$\gamma_{k_3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.5</td>
<td>0.5</td>
<td>9.5</td>
<td>0.5</td>
</tr>
<tr>
<td>2</td>
<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
<td>9.5</td>
</tr>
<tr>
<td>3</td>
<td>0.5</td>
<td>0.5</td>
<td>4.5</td>
<td>4.5</td>
</tr>
<tr>
<td>$p_\kappa$</td>
<td>0.99</td>
<td>0.005</td>
<td>0.005</td>
<td></td>
</tr>
</tbody>
</table>

Figure 4: EVA based on volumes $u_1$ and $u_2$ and for fixed $u_3 = 1$. Point A reflects $u_1 = u_2 = 1$; points B and B’ are optimal based on true risk distribution.

6 Conclusion

This paper demonstrates that the classical approaches for calibrating the matrix $R$—specifically the assumption that it has ones on the diagonal—can make it impossible to
fit the square-root formula to a true multivariate distribution with regard to first and second-order derivatives of the capital requirement with respect to exposure changes. We thereby identify the root cause of moral hazard effects of an improper risk aggregation, as empirically identified by Chen et al. (2019). In connection with the partial-derivative implied correlations, which we propose in section 4, the square-root formula overcomes the moral hazard effects problem.

We conclude with two remarks. Firstly, we believe that it is important that the square-root formula correctly measures marginal effects of exposure changes. The square-root formula is thereby compatible with Euler capital allocation, which can give the directions of reasonable portfolio changes; moreover, one can correctly characterize a (locally) optimal portfolio. As outlined by Gründl and Schmeiser (2007), capital allocation is not helpful to assess large portfolio changes. In the context of Solvency II, insurers should not fully rely on the standard formula to assess large portfolio changes, but assess the new risk based on a variety of appropriate methods in a so called “Own Risk and Solvency Assessment” (ORSA). Secondly, we have assumed throughout the paper that the true risk distribution is known at the two stages of calculations: at stage one, the calibration of the matrix \( R \) makes use of the true multivariate distribution and at stage two, the calculation of the aggregate capital requirement employs the true univariate capital requirements, which are collected in the vector \( x \). Hence, the biases that we measure in section 3 can be fully attributed to the calibration approach, and they are not affected by sampling error or the choice of an estimator. When thinking about an estimator for \( R \) under partial information, it has been unclear up to now how the bias of such an estimator could be assessed, given that there is leeway in choosing the matrix \( R \) which reflects the true aggregate capital requirement when dealing with \( n \geq 3 \) risks. Deriving
a proper estimator of the partial-derivative implied correlations is an important question for future research. A starting point is offered by the use of mixed-gamma distribution in connection with an approach of estimating its parameters as suggested by Furman et al. (2019, p. 19 f.).
Appendix

A Proof of Proposition 1

For each $s \in \{1, \ldots, n(n-1)/2 - 1\}$, choose the indices $1 \leq k_s < \ell_s \leq n$ and define a portfolio $w^{(s)} \in \mathbb{R}^n$ with $w^{(s)}_{k_s} = w^{(s)}_{\ell_s} = 1$ and $w^{(s)}_i = 0$ for all other entries $i$. The indices shall be chosen such that the $n(n-1)/2 - 1$ portfolios are pairwise different. The last portfolio’s entries are set to $w^{(n(n-1)/2)}_k = 1$ for all $k = 1, \ldots, n$. For $s \in \{1, \ldots, n(n-1)/2\}$, we set up the equation

$$\sum_{i=1}^n \sum_{j=1}^n \varrho_{ij} \text{CapR}_{1-\alpha}(w^{(s)}_i X_i) \text{CapR}_{1-\alpha}(w^{(s)}_j X_j) = \text{CapR}_{1-\alpha}\left(\sum_{i=1}^n w^{(s)}_i X_i\right)^2,$$

which is linear in the variables $\varrho_{ij}$. The first $n(n-1)/2 - 1$ equations simplify to

$$2\varrho_{k_s\ell_s} \text{CapR}_{1-\alpha}(X_{k_s}) \text{CapR}_{1-\alpha}(X_{\ell_s}) = \text{CapR}_{1-\alpha}\left(X_{k_s} + X_{\ell_s}\right)^2$$

and can be uniquely solved for $\varrho_{k_s\ell_s}$. The last equation simplifies to

$$\sum_{i=1}^n \sum_{j=1}^n \varrho_{ij} \text{CapR}_{1-\alpha}(X_i) \text{CapR}_{1-\alpha}(X_j) = \text{CapR}_{1-\alpha}\left(\sum_{i=1}^n X_i\right)^2 \quad (43)$$

and it can be solved for the last unknown parameter $\varrho_{k\ell}$. Taking the square-root on both sides of Eq. (43) implies $\sqrt{x^T R x} = \text{CapR}_{1-\alpha}(X)$. 

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B  Proof of Theorem 1

We write $\partial_\ell = \partial/\partial x_\ell$ and $\partial_\ell = \partial/\partial u_\ell$, applying to functions of the variable $x$, or $u$, respectively. Note that the chain rule implies

$$\partial_\ell f^2(x) = 2f(x)\partial_\ell f(x) \quad (44)$$

and

$$\varrho_{k\ell}(x) = \frac{1}{2} \frac{\partial^2}{\partial x_k \partial x_\ell} f^2(x) = \frac{\partial}{\partial x_\ell} \left\{ f(x) \frac{\partial f}{\partial x_k}(x) \right\} = \partial_\ell \{ f(x) \partial_k f(x) \} \quad (45)$$

Schwarz’s Theorem on the symmetry of second-order derivatives shows the symmetry of $R$, and the product rule implies

$$\partial_\ell \left\{ f(x) \sum_{k=1}^n x_k \partial_k f(x) \right\} = \partial_\ell \left\{ \sum_{k=1}^n x_k (f(x) \partial_k f(x)) \right\} = \sum_{k=1}^n x_k \partial_\ell \{ f(x) \partial_k f(x) \} + f(x) \partial_\ell f(x) \quad (46)$$

We use this and Euler’s Theorem on homogeneous functions (cf. Tasche, 2008, p. 4), namely

$$\sum_{\ell=1}^n x_\ell \partial_\ell f(x) = f(x), \quad (47)$$

and derive

$$(gx(1_n))^2 = \sum_{k=1}^n \sum_{\ell=1}^n x_k x_\ell \varrho_{k\ell} \overset{(45)}{=} \sum_{k=1}^n \sum_{\ell=1}^n x_k x_\ell \partial_\ell \{ f(x) \partial_k f(x) \} = \sum_{\ell=1}^n x_\ell \left[ \sum_{k=1}^n x_k \partial_\ell \{ f(x) \partial_k f(x) \} \right]$$

$$\overset{(46)}{=} \sum_{\ell=1}^n x_\ell \left[ \partial_\ell \left\{ f(x) \sum_{k=1}^n x_k \partial_k f(x) \right\} - f(x) \partial_\ell f(x) \right]$$

$$\overset{(47)}{=} \sum_{\ell=1}^n x_\ell \left[ \partial_\ell f^2(x) - f(x) \partial_\ell f(x) \right] \overset{(44)}{=} \sum_{\ell=1}^n f(x) x_\ell \partial_\ell f(x) \overset{(47)}{=} f^2(x) = (f_x(1_n))^2$$
The assumption $f(x) > 0$ now implies (28). To prove equation (29), we note that the chain rule and (26) imply

$$\partial_t f_x(1_n) = x_t \partial_t f(x)$$  \hspace{1cm} (48)

We derive

$$\partial_t g_x(1_n)^{(10)} = \frac{x_t}{g_x(1_n)} \sum_{k=1}^{n} \partial_k g_k x_k = \frac{x_t}{f_x(1_n)} \sum_{k=1}^{n} x_k \partial_t \{f(x)\partial_k f(x)\}$$

$$= \frac{x_t}{f_x(1_n)} \left[ \partial_t \left\{ f(x) \sum_{k=1}^{n} x_k \partial_k f(x) \right\} - f(x)\partial_t f(x) \right]$$

$$= \frac{x_t}{f_x(1_n)} \left[ \partial_t f^2(x) - f(x)\partial_t f(x) \right] = \frac{x_t f(x) \partial_t f(x)}{f_x(1_n)} = \partial_t f_x(1_n)$$

This is equation (29). To prove Equation (30), we write $\partial_{k\ell} = \partial^2 / (\partial x_k \partial x_\ell)$ for functions of $x$ and $\partial_{k\ell} = \partial^2 / \partial u_k \partial u_\ell$ for functions of $u$. The chain rule and (26) imply

$$\partial_{k\ell} f^2_x(1_n) = x_k x_\ell \partial_{k\ell} f^2(x)$$  \hspace{1cm} (49)

We derive

$$\partial_{k\ell} g^2_x(1_n) = 2 x_k x_\ell \partial_{k\ell} = 2 x_k x_\ell \partial_t \{f(x)\partial_k f(x)\} = x_k x_\ell \partial_{k\ell} f^2(x) = \partial_{k\ell} f^2_x(1_n)$$  \hspace{1cm} (50)

We thereby have

$$\frac{1}{2} \partial_{k\ell} g^2_x(1_n) = \frac{1}{2} \partial_{k\ell} f^2_x(1_n)$$

$$\iff \partial_t (g_x(1_n) \cdot \partial_k g_x(1_n)) = \partial_t (f_x(1_n) \cdot \partial_k f_x(1_n))$$

$$\iff \partial_t g_x(1_n) \partial_k g_x(1_n) + g_x(1_n) \partial_t \partial_k g_x(1_n) = \partial_t f_x(1_n) \partial_k f_x(1_n) + f_x(1_n) \partial_{k\ell} f_x(1_n)$$

$$(28), (29) \iff \partial_{k\ell} g_x(1_n) = \partial_{k\ell} f_x(1_n)$$
C Proof of Proposition 2

Let the entries of the vector $\bar{x} \in \mathbb{R}^n$ be defined as $\bar{x}_i = x_i^{-1}$. There is an open set $U \subseteq \mathbb{R}^n$ such that $x \in U$ and we can define the function $f : U \rightarrow \mathbb{R}$ as

$$s \rightarrow f(s) = f_X(s \circ \bar{x}),$$

which is homogeneous of degree one and twice continuously differentiable on $U$. For all $u \in M$, the right-hand sides of (27) and (34), each in connection with the vector $x$ defined by (32), coincide. To prove this, we show that the entries of underlying matrices $R$ coincide. We start with the definition in (25):

$$\frac{1}{2} \frac{\partial^2}{\partial s_k \partial s_\ell} f^2(s) \bigg|_{s=x} = \frac{1}{2} \frac{\partial^2}{\partial s_k \partial s_\ell} f_X^2(s \circ \bar{x}) \bigg|_{s=x} = \frac{1}{2} \frac{\partial^2}{\partial u_k \partial u_\ell} f_X^2(1_n)$$

Moreover, for all $u \in M$, the function $f_x(u)$ defined in (26) in connection with the vector $x$ defined by (32) coincides with $f_X(u)$:

$$f_x(u) \overset{(26)}{=} f(u \circ x) \overset{(51)}{=} f_X(u \circ x \circ \bar{x}) = f_X(u)$$

Thereby, Theorem 1 implies all statements of Proposition 2.
D Example based on the Gamma distribution

Incorporating the matrix in (38) into the square-root formula implies that the marginal capital requirements are (cf. Eq. (12))

\[
\frac{(Rx) \circ x}{\sqrt{x^T R x}} = \frac{1}{7.0565} \begin{pmatrix}
1.0244 & -0.0824 \\
-0.0824 & 0.5958
\end{pmatrix} \begin{pmatrix}
6.879 \\
2.715
\end{pmatrix} \circ \begin{pmatrix}
6.879 \\
2.715
\end{pmatrix} = \begin{pmatrix}
6.6523 \\
0.4042
\end{pmatrix}
\]

Moreover, the second-order derivatives of the capital requirement according to the square-root formula are obtained by Eq. (13) as

\[
\frac{\partial^2}{\partial u_1 \partial u_1} g_x(u) = \frac{1.0244 \cdot 6.879^2 - 6.6523^2}{7.0565} = 0.5993
\]
\[
\frac{\partial^2}{\partial u_1 \partial u_2} g_x(u) = \frac{(-0.0824) \cdot 6.879 \cdot 2.715 - 6.6523 \cdot 0.4042}{7.0565} = -0.5993
\]
\[
\frac{\partial^2}{\partial u_2 \partial u_2} g_x(u) = \frac{0.5958 \cdot 2.715^2 - 0.4042^2}{7.0565} = 0.5993
\]

As stated by Theorem 1, the just calculated marginal capital requirements and second-order derivatives based on the square-root formula coincide with the results when all calculations are based on the true risk distribution.
References


