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Scenario-based Capital Requirements for the Interest Rate Risk of Insurance Companies

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Abstract

Insurance companies, particularly life insurers, can suffer substantially from changing interest rates. Regulatory approaches, such as the Solvency-II-standard formula, measure interest rate risk based on scenarios, which are easier to implement and more comprehensible than the results of a simulation-based internal model. Backtesting the standard formula scenarios against historical yield curve movements indicates that they are too optimistic. Also, the standard formula focuses on shifts in the yield curve, but neglects changes in its steepness or curvature. If further scenarios were to be added, the question arises of how to aggregate the outcomes towards a Value-at-Risk figure. This paper starts from a stochastic model for interest rates, which builds on the dynamic version of the Nelson-Siegel model. The latter is modified such that it respects a lower bound for interest rates. We then use a principal component analysis to translate the simulated yield curves into scenarios. The Value-at-Risk for interest rate risk can be derived by aggregating the changes in equity capital in each scenario similarly to the “square-root formula” of the Solvency-II-standard formula. Backtesting results indicate that four scenarios suffice to measure interest rate risk in accordance with historical yield curve movements and almost as exactly as a stochastic model.

Keywords: Value-at-Risk, Interest Rate Risk, Principal Component Analysis, Solvency II
JEL classification: G17, G22, G28

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1 Introduction

Interest rate risk is one of the most important risks for insurance companies. Since insurers face long-term obligations, they invest over long time horizons, and a large portion of their assets are fixed income investments such as bonds or mortgage loans. Typically, the durations of assets and liabilities are not matched, but life insurers attain much longer durations on their liability side than on their asset side.\footnote{EIOPA (2014a, p. 17) report that durations of liabilities are on average 10 years longer than those of assets for Austrian, German, Lithuanian and Swedish insurers. Möhlmann (2017) uses accounting data of German life insurers and estimates that their modified duration gap is 4.9 when weighting by the size of insurance companies. The unweighted estimate is 6.8, indicating that “smaller insurers tend to have a wider duration gap” (cf. Möhlmann, 2017, p. 10).} Modern insurance regulation frameworks, such as Solvency II in the European Economic Area, impose risk-based capital requirements to address those risks. Under Solvency II, the capital requirement is defined as the 99.5\% Value-at-Risk of the change in economic capital over one year.\footnote{Cf. European Commission (2009), Art. 101 (3).} Most insurers calculate the capital requirement with the standard formula. Regarding interest rate risk, the standard formula applies multiplicative stress factors to the current yield curve to determine an upward and a downward movement of the yield curve.\footnote{The scenarios are defined in European Commission (2015), Articles 166 f.} Insurers need to recalculate their capital in these two scenarios and obtain their capital requirement for interest rate risk as the maximal loss in capital that can result from the two scenarios.

This procedure is questionable, particularly in three respects. Firstly, the calibration of the stress factors, at least for the downward scenario, appears much too optimistic. Between 1999 and 2015, the downward stress scenario underestimated the drop in interest rates during the subsequent 12 months for periods in 2011 as well as between 2014 and 2015 (cf. EIOPA, 2016, p. 59). This indicates that the scenarios do not reflect the 1-in-
200-year event, which would correspond to the 99.5% Value-at-Risk. In addition to the poor backtesting result, Gatzert and Martin (2012) and Braun et al. (2017) demonstrate deficiencies of the standard formula’s market risk assessment when comparing it against a partial internal model. Secondly, the standard formula systematically underestimates the risk from changes in the steepness and/or curvature of the yield curve. Since both stress scenarios reflect yield curve shifts, an insurer can immunize against them by closing the duration gap, meaning that the capital requirement could drop to zero. However, zero certainly underestimates the insurer’s interest rate risk, since changes in the steepness or curvature of the yield curve can still lead to losses. While the second point of criticism could be solved by increasing the number of scenarios, the third point refers to a lack of theoretical foundation for the concept of aggregating scenario outcomes towards a Value-at-Risk figure. This point of criticism refers not only to the Solvency II standard formula, but to the measurement of interest rate risk based on stress scenarios in general. For instance, the Bank for International Settlements (2016, p. 8) proposes that banks should assess their interest rate risk by calculating “the impact on economic value and earnings of multiple scenarios”. In the context of identifying outlier banks, the maximal loss in capital according to six prescribed scenarios is relevant. Whether this procedure can provide a consistent estimate for the Value-at-Risk (or any other useful risk measure) seems doubtful.

The objective of this article is to derive a methodology for measuring interest rate risk in line with the following requirements:

\[\text{Classically, closing the duration gap immunizes a portfolio against parallel shifts in the yield curve. Given that the yield curve scenarios in the standard formula are not parallel shifts, one could implement the more general approach of Litterman and Scheinkman (1991, p. 55 f.), who explain how to construct a portfolio that is immunized against a particular movement (not necessarily a parallel shift) of the yield curve.}\]
1. In light of Solvency II regulations, the method shall allow for estimating the Value-at-Risk. In particular, the 99.5% Value-at-Risk with a 1-year holding period is of interest.

2. The measurement shall reflect an insurer’s exposure to changes in the level, in the steepness and in the curvature of the yield curve.

3. The measurement shall be in accordance with losses that (at least typical) insurance companies would have realized for historical yield curve movements.

4. It shall take into account that there is a lower bound for interest rates reflecting the economic costs of storing cash.

5. It shall be a pragmatic approach which insurers can implement for example by recalculating their capital in a small number of scenarios.

The first part of this paper is dedicated to deriving meaningful simulations of yield curves in light of the requirements 1-4. To this end, we start from a dynamic version of the Nelson and Siegel (1987) model, which represents the yield curve based on four parameters. Diebold and Li (2006) derive good forecasts of future yield curves by modeling the development of these parameters over time with stochastic processes. In particular, the forecasts outperform those of affine factor models, which we therefore do not use.\textsuperscript{6} Caldeira et al. (2015) demonstrate that a stochastic calibration of the Dynamic Nelson-

\textsuperscript{6}The weak performance of affine factor models in forecasting the yield curve has also been highlighted by Duffee (2002). Moreover, by modeling the short rate, affine factor models focus on yield curve shifts and may therefore understate the risk of changes in the steepness or the curvature of the yield curve. Vedani et al. (2017) point out that an insurance-specific version of the LIBOR Market Model, which is often employed by practitioners to valuate options and guarantees embedded in insurance liabilities, leads to spurious simulated yield curves when the model is applied for longer time horizons.
Siegel (DNS) model provides a good basis for estimating the Value-at-Risk of fixed income portfolios over a 1-day holding period.

The DNS model, as applied by Caldeira et al. (2015), does not account for lower bounds for interest rates. For longer holding periods, in particular when being situated in a low yield environment, it can simulate high negative interest rates, which are not reasonable from an economic perspective. To solve this issue, we suggest using the DNS model for modeling the logarithmic difference between interest rates and their lower bound.7 We call this variant of the model the “Log-DNS” model.

We backtest the Value-at-Risk estimates according to the DNS model and the Log-DNS model for various combinations of confidence levels and holding periods against historical yield curve changes. The backtesting is conducted for 1,000 hypothetic asset-liability portfolios. The set of these portfolios has been composed such that it reflects the empirical findings by Möhlmann (2017) for German life insurers.8 For each portfolio, the accuracy of the Value-at-Risk is measured by the portion of historical time windows for which the Value-at-Risk is lower than the loss in value that the portfolio experienced for the actual change in interest rates (hit rate).

The backtesting results demonstrate that the accuracy of the Value-at-Risk estimates provided by the Log-DNS model is similar to those of the DNS model. Hence, the Log-DNS model’s advantage of complying with a lower bound for interest rates is not dampened by deficiencies in terms of the accuracy. Moreover, we find that the Value-at-

7An alternative approach for handling the issue is proposed by Eder et al. (2014), who incorporate a lower bound for interest rates by means of a plane-truncated normal distribution. However, the approach is numerically extensive, and it therefore seems difficult to combine it with a GARCH process to address heteroscedasticity in longer time horizons.
8Due to requirement 3, the backtesting is more challenging than that performed by Caldeira et al. (2015, p. 72), who focus on equally-weighted asset portfolios.
Risk according to both DNS and Log-DNS models is suitable for most of the asset-liability portfolios considered, i.e., not only for an equally-weighted asset portfolio as considered by Caldeira et al. (2015).

In the second part of the paper, we derive a scenario-based approximation of the Value-at-Risk in order to address requirement 5. The scenarios are elicited by a principal component analysis from the simulated yield curves according to the Log-DNS model; therefore, the scenarios respect lower bounds for interest rates as well. To aggregate the scenario outcomes towards an approximate Value-at-Risk figure, a “square-root formula” is applied, analogous to the aggregation of (sub)modules in the Solvency-II-standard formula. Given that the principal component scores are by construction uncorrelated, one might think that correlation parameters can be left out in this aggregation. However, as pointed out by Campbell et al. (2008) as well as EIOPA (2014b, p. 8), the correlation parameters not only reflect classical Pearson correlations, but can also compensate inaccuracies in the aggregation resulting from skewed or fat-tailed distributions. In this sense, we propose allowing for a small number of correlation parameters when aggregating the scenario outcomes. Our backtesting results for the scenario-based assessment demonstrate that a calculation based on four scenarios in connection with two correlation parameters provides a close approximation of simulation-based Value-at-Risk.

The remainder of the paper is organized as follows. Section 2 outlines the methodology for stochastically modeling interest rate risk, determining the Value-at-Risk and transforming the simulated yield curves into a scenario-based calculation for the Value-at-Risk. Section 3 calibrates the models based on yield curve data published by the European Central Bank (ECB). Section 4 provides the backtesting of the stochastic models as well as of the scenario-based approximation. Section 5 concludes.
2 Value-at-Risk for interest rate risk

2.1 Firm model

We consider an insurance company that expects future cash inflows $A_1, \ldots, A_M \geq 0$ from assets and outflows $L_1, \ldots, L_M \geq 0$ from insurance obligations in 1, 2, ..., $M$ years, where $M$ denotes the largest maturity under consideration. The expected surpluses, $S_\tau = A_\tau - L_\tau$ for $\tau = 1, \ldots, M$, are collected in a column vector $S = (S_1, \ldots, S_M)'$. The firm’s economic equity capital (i.e. the interest-rate-sensitive part of it) at time 0 (the balance sheet date) is obtained as the present value of the surpluses:

$$E_0 = \sum_{\tau = 1}^{M} e^{-r_0(\tau)\cdot \tau} S_\tau,$$

where $r_0(\tau)$ is the continuously compounded risk-free interest rate for maturity $\tau$ at time 0.

In line with Solvency II regulations, interest rate risk is measured based on the loss in equity capital caused by an instantaneous change in interest rates.\(^\text{9}\) If interest rates change instantaneously from $r_0(\tau)$ to $\tilde{r}(\tau)$, the firm’s equity capital changes to

$$\tilde{E}_0 = \sum_{\tau = 1}^{M} e^{-\tau \cdot \tilde{r}(\tau)} S_\tau.$$

Understanding the interest rates $(\tilde{r}(\tau))_{\tau \in \{1, \ldots, M\}}$ as a random vector, $\tilde{E}_0$ is also a random variable. In order to determine a Value-at-Risk for a specified holding period, we comprehend the vector of interest rates $(r_h(\tau))_{\tau \in \{1, \ldots, M\}}$ as a multivariate stochastic process.

\(^{9}\)Cf. European Commission (2015), articles 166 f.
over time $h$. Analogously to Eq. 2, the firm’s equity capital when interest rates have changed instantaneously from $r_0(\tau)$ to $r_h(\tau)$ is defined as

$$E_{0,h} = \sum_{\tau=1}^{M} e^{-\tau r_h(\tau)} S_\tau,$$

and the Value-at-Risk for interest rate risk with confidence level $1-\alpha$ and holding period $h$ is obtained as

$$\text{VaR}_{1-\alpha,h} = -q_\alpha(E_{0,h} - E_0),$$

where $q_\alpha(X)$ denotes the $\alpha$-quantile of the random variable $X$. The Solvency II capital requirement for interest rate risk is determined as $\text{VaR}_{99.5\%,1 \text{ year}}$.

### 2.2 Modeling interest rate risk

The starting point for modeling interest rates is the result of Caldeira et al. (2015), who find that the dynamic version of the model from Nelson and Siegel (1987) provides a good basis for determining the Value-at-Risk for interest rate risk. In this sense, we model the continuously compounded interest rates for a set of maturities $\tau \in \{\tau_1, \ldots, \tau_m\} \subset \{1, \ldots, M\}$ at time $t$ as

$$r_t = \Lambda(\lambda, \tau) f_t + \epsilon_t,$$

where $\Lambda(\lambda, \tau)$ is a $m \times 3$ matrix of factor loadings, $f_t$ is a 3-dimensional stochastic process of factor scores and $\epsilon_t$ is an $m$-dimensional stochastic process of disturbances. According
to Diebold and Li (2006, p. 341), each row \( i \in \{1, \ldots, m\} \) of the matrix of factor loadings \( \Lambda(\lambda, \tau) \) is defined as

\[
\begin{bmatrix}
1, & 1 - e^{-\tau_i/\lambda}, & 1 - e^{-\tau_i/\lambda} - e^{-\tau_i/\lambda}
\end{bmatrix}
\]

(6)

The components of the vector of factors scores \( f_t = (\beta_{1,t}, \beta_{2,t}, \beta_{3,t})' \) can be intuitively interpreted. \(^{10}\) \( \beta_{1,t} \) reflects the long-term level of the yield curve, since its loading is constantly 1. The loading on \( \beta_{3,t} \) starts at 1 if \( \tau \) is close to zero and decreases to zero if \( \tau \) becomes large. Hence, \( \beta_{2,t} \) can be viewed as the short-term interest rate and it governs the slope of the yield curve. The loading on \( \beta_{3,t} \) starts at zero, becomes positive and finally converges to zero if \( \tau \) moves from 0 to infinity. Hence, \( \beta_{3,t} \) steers medium-term interest rates, or the curvature of the yield curve.

A stochastic calibration of the dynamic Nelson-Siegel (DNS) model in Eq. 5 will serve as a benchmark model for our analyses. In order to avoid the modeled interest rates falling below an economically reasonable level, we consider an additional specification of the model, which incorporates a lower bound for interest rates. The right-hand side of Eq. 5 is now used to model the logarithmic difference between the interest rate and a lower bound for the interest rate:

\[
\ln(r_t - r_{\text{min}}) = \Lambda(\lambda, \tau)f_t + \epsilon_t,
\]

(7)

where \( r_{\text{min}} = (r_{\text{min}}(\tau_1), \ldots, r_{\text{min}}(\tau_m))' \) is the vector of lower bounds per maturity \( \tau_i \), the logarithm is applied to every entry of the vector \( (r_t - r_{\text{min}}) \) separately, and \( \Lambda(\lambda, \tau) \) is defined as in Eq. 6. As for the model in Eq. 5, the factor scores \( f_t = (\beta_{1,t}, \beta_{2,t}, \beta_{3,t})' \)

\(^{10}\) Cf. Diebold and Li (2006, p. 341 f.).
govern the level, slope and curvature of the yield curve. We refer to this model as the Log-DNS model.

For the DNS and Log-DNS models, the development of the parameter vector \( f_t \) over time may exhibit autocorrelation. We address autocorrelation by a vector-autoregressive (VAR) model, in which the development of each entry \( f_t \) depends on the history of all entries of \( f_t \):

\[
\Delta f_t = \mu + \sum_{k=1}^{p} \Gamma_k \cdot f_{t-k} + \eta_t, \tag{8}
\]

where \( \Delta f_t = f_t - f_{t-1} \), \( p \in \mathbb{N} \) is the lag order of the VAR process, \( \mu \in \mathbb{R}^3 \) is a vector of constant coefficients, \( \Gamma_k \) are \( 3 \times 3 \) transition matrices, and the 3-dimensional stochastic process \( \eta_t \) reflects the disturbances.

The disturbances process \( \eta_t \) may exhibit time-varying volatilities and correlations. The backtesting results of Caldeira et al. (2015, p. 77-79) indicate that the dynamic conditional correlation (DCC) model proposed by Engle (2002) is appropriate to model the disturbances. In this model, the covariance matrix \( \Omega_t \) is decomposed into a time-varying correlation matrix \( R_t \) and a \( 3 \times 3 \) diagonal matrix \( D_t \) such that

\[
\Omega_t = D_t R_t D_t. \tag{9}
\]

Using

\[
z_t = D_t^{-1} \eta_t, \tag{10}
\]

\[11\]As an alternative to the VAR model, one could describe the development of each entry of \( f_t \) separately using an autoregressive (AR) model. Since the backtesting results of Caldeira et al. (2015, p. 77-79) demonstrate that the VAR model works better than a combination of AR models, we omit the latter specification.
the $\eta_t$ are transformed into $(3 \times 1)$-vectors $z_t$ of uncorrelated, standardized disturbances with mean zero and variance one. The elements in the correlation matrices $R_t$ are denoted by $\rho_{i,j,t}$ and obtained as

$$\rho_{i,j,t} = \frac{q_{i,j,t}}{\sqrt{q_{i,i,t}q_{j,j,t}}},$$

where the $q_{i,j,t}$ are the elements of $3 \times 3$ matrices $Q_t$. The diagonal matrix $D_t$ and the matrix $Q_t$ follow GARCH-like processes:

$$D_t^2 = \text{diag}(\omega_i) + \text{diag}(\kappa_i) \circ \eta_{t-1} \eta_{t-1}' + \text{diag}(\lambda_i) \circ D_{t-1}^2$$

$$Q_t = (1-a-b)\bar{Q} + a z_{t-1} z_{t-1}' + b Q_{t-1}$$

where $\text{diag}(x_i)$ generates a $3 \times 3$ diagonal matrix with $x_1, x_2, x_3$ on the diagonal, $\circ$ denotes the Hadamard product, $\bar{Q}$ is the unconditional covariance matrix, $\omega_i, \kappa_i, \lambda_i$ are non-negative parameters $\forall i \in \{1, 2, 3\}$, and $a, b$ are non-negative parameters such that $a + b < 1$.

Finally, the residuals $\epsilon_t$ of the DNS and Log-DNS models may exhibit autocorrelation. We address this by modeling the residuals for maturities $\tau_1, ..., \tau_m$ by means of autoregressive processes with lag order 1. The disturbances of these AR(1) processes are modeled by independent normal distributions.

In total, both models can be used to simulate interest rates at a future point in time for all maturities of interest, $\tau \in \{1, ..., M\}$. To extrapolate the simulated interest rates for maturities $\tau_1, ..., \tau_m$ towards interest rates for all maturities $1, ..., M$, the model of Svensson (1994) is used. The Svensson model is regularly employed by central banks to
elicit a yield curve out of bond market data.\textsuperscript{12} It extends the Nelson-Siegel model by an additional factor and determines interest rates as $r_t(\tau) = \Lambda(\lambda_1, \lambda_2, \tau)f_t$, where each row $i \in \{1, \ldots, N\}$ of the matrix of factor loadings $\Lambda(\lambda_1, \lambda_2, \tau)$ is defined as

$$
\begin{bmatrix}
1, & 1 - e^{-\tau_i/\lambda_1} / \tau_i/\lambda_1, & 1 - e^{-\tau_i/\lambda_1}, & 1 - e^{-\tau_i/\lambda_2}, & 1 - e^{-\tau_i/\lambda_2} - e^{-\tau_i/\lambda_2} & \\
1 - e^{-\tau_i/\lambda_1}, & 1 - e^{-\tau_i/\lambda_1} / \tau_i/\lambda_1 - e^{-\tau_i/\lambda_2} / \tau_i/\lambda_2 - e^{-\tau_i/\lambda_2} & \\
1 - e^{-\tau_i/\lambda_1} / \tau_i/\lambda_1 & & & & & \\
1 - e^{-\tau_i/\lambda_1} & & & & & \\
1 - e^{-\tau_i/\lambda_2} / \tau_i/\lambda_2 - e^{-\tau_i/\lambda_2} & & & & & \\
1 - e^{-\tau_i/\lambda_2} & & & & & \\
\end{bmatrix}
$$

(14)

Using the Svensson model for the extrapolation allows us to model the residuals $\epsilon_t$ only for a small set of maturities $\{\tau_1, \ldots, \tau_m\}$ and receive a meaningful yield curve for every simulation path of the stochastic model.

### 2.3 Scenario-based Value-at-Risk

The models in section 2.2 can be used to generate a large number of simulated yield curves for a future point in time (e.g. in one year), which can be directly used to determine the Value-at-Risk according to Eq. 16. For a standard formula, however, this procedure might not be appropriate, since complex information (i.e. the modeled yield curve in a large number of simulations) would need to be reported by the regulator and the recalculations of assets and liabilities by the insurers would be extensive. The aim of this section is to approximate the Value-at-Risk with a simplified calculation method, in order to reduce the information that the regulator needs to provide to a small number of scenarios.

As a starting point we assume that the portfolio losses are linear in the discount factors;\textsuperscript{13} hence, the discount factors are the actual risk drivers. Consider an insurance company with expected surpluses $S = (S_{\tau_1}, \ldots, S_{\tau_K})'$ at maturities $\tau_1, \ldots, \tau_K$ and let $X_1$ denote

\textsuperscript{12}For instance, the yield data used for the calibration later on in section 3.1 have been obtained by the European Central Bank (ECB) based on the Svensson model.

\textsuperscript{13}This assumption is discussed later on in section 5.
the random vector with the discount factors corresponding to interest rates for those maturities in 1 year:

\[ X_1 = (e^{-r_1 \text{ year}()}, ..., e^{-r_K \text{ year}()} )', \] (15)

Moreover, let \( E(X_1) \) denote its expectation and \( X_0 \) the corresponding deterministic vector of discount factors at time 0. Then, the Value-at-Risk for interest rate risk is obtained as

\[ \text{VaR}_{1-\alpha, \text{ year}} = -q_{\alpha}(X'_1 \cdot S) - X'_0 \cdot S \] (16)

In order to reduce the required information for this calculation, we transform \( X_1 \) into its principal components such that

\[ X_1 = \Theta \cdot Y + E[X_1] \] (17)

By construction of the principal component analysis (PCA), the vector of scores, \( Y \), is a random vector of order \( K \) with \( E[Y] = 0 \), the covariance matrix of which is a diagonal matrix. We can recalculate the Value-at-Risk in line 16 as

\[ -q_{\alpha}((\Theta \cdot Y + E[X_1])' \cdot S) - X'_0 \cdot S \]

\[ = -q_{\alpha}((\Theta \cdot Y)' \cdot S) - (E[X_1] - X_0)' \cdot S \]

\[ = -q_{\alpha}\left( \sum_{k=1}^{K} Y_k \cdot (\Theta' \cdot S)_k \right) - (E[X_1] - X_0)' \cdot S \] (18)

Here, \((\Theta' \cdot S)_k\) is the \( k \)-th entry of the vector \((\Theta' \cdot S)\) and reflects the insurer’s exposure to the \( k \)-th principal component. Let us assume for a moment that \( X_1 \) (and hence \( Y \)) follow a multivariate elliptical distribution, and let us denote the \( \alpha \)-percentile of the
standardized marginal distribution by $z_\alpha$. Then, the Value-at-Risk in line 18 can be determined by

$$\sqrt{\text{var} \left( \sum_{k=1}^{K} Y_k \cdot (\Theta' \cdot S)_k \right) \cdot z_\alpha - (\mathbb{E}[X_1] - X_0)' \cdot S},$$

(19)

where $\text{var}(X)$ denotes the variance of $X$. Since the covariance matrix of $Y$ is diagonal, line 19 can be rewritten as

$$\sqrt{\sum_{k=1}^{K} \text{var}(Y_k) \cdot (\Theta' \cdot S)_k^2 \cdot z_\alpha^2 - (\mathbb{E}[X_1] - X_0)' \cdot S}$$

(20)

According to the assumption of an elliptical distribution, we have

$$\text{var}(Y_k) \cdot (\Theta' \cdot S)_k^2 \cdot z_\alpha^2 = \left( q_\alpha \left( Y_k \cdot (\Theta' \cdot S)_k \right) \right)^2$$

(21)

The quantile on the right-hand side of Eq. 21 measures the risk related to principal component $k$, which we denote by $\text{VaR}_k$. Irrespective of the distribution assumption for $Y$, we can rewrite $\text{VaR}_k$ by pulling out the factor $(\Theta' \cdot S)_k$:

$$\text{VaR}_k = \begin{cases} 
q_\alpha(Y_k) \cdot (\Theta' \cdot S)_k & \text{if } (\Theta' \cdot S)_k \geq 0 \\
q_{1-\alpha}(Y_k) \cdot (\Theta' \cdot S)_k & \text{if } (\Theta' \cdot S)_k < 0 
\end{cases}$$

$$= \begin{cases} 
\sum_{\tau=1}^{K} \left( X_{0,\tau} + q_\alpha(Y_k) \cdot \Theta_{\tau,k} \right) \cdot S_{\tau} - X_0 \cdot S & \text{if } (\Theta' \cdot S)_k \geq 0 \\
\sum_{\tau=1}^{K} \left( X_{0,\tau} + q_{1-\alpha}(Y_k) \cdot \Theta_{\tau,k} \right) \cdot S_{\tau} - X_0 \cdot S & \text{if } (\Theta' \cdot S)_k < 0 
\end{cases}$$

(22)
where $X_{0,\tau} = e^{-\tau_0(\tau)}$ denotes the $\tau$-th element of $X_0$. The expressions $X_{0,\tau} + q_\alpha(Y_k) \cdot \Theta_{\tau,k}$ and $X_{0,\tau} + q_{1-\alpha}(Y_k) \cdot \Theta_{\tau,k}$ in line 22 can be comprehended as discount factors for maturity $\tau$ years and related to principal component $k$. Based on

$$r^{k,1}(\tau) = -\ln[X_{0,\tau} + q_\alpha(Y_k) \cdot \Theta_{\tau,k}] / \tau$$

(23)

and

$$r^{k,2}(\tau) = -\ln[X_{0,\tau} + q_{1-\alpha}(Y_k) \cdot \Theta_{\tau,k}] / \tau,$$

(24)

$\tau = \tau_1, ..., \tau_K$, they can be translated into “stressed” interest rates related to that principal component. Hence, the Value-at-Risk related to principal component $k$ is calculated as the change in equity capital when the yield curve changes in a stress scenario.

In order to receive the Value-at-Risk for interest rate risk in total, the results for $\text{VaR}_k$ are aggregated as in Eq. 20. Since the first few components typically explain a large share of the variation, a good approximation of the Value-at-Risk might already be achieved by taking only the first $\tilde{K} < K$ components into account:

$$\text{VaR}_{1-\alpha}(1\text{ year}) \approx \sqrt{\sum_{k=1}^{\tilde{K}} \text{VaR}_k^2 - \left( E[X_1] - X_0 \right)'} \cdot S$$

(25)

An appropriate value for $\tilde{K}$ needs to trade off the benefits of a higher accuracy against the costs of a more complex calculation, since more scenarios need to be evaluated.

### 2.4 Scenario-based Value-at-Risk with correlation parameters

As an alternative to increasing the number of scenarios, there is a more effective possibility for improving the accuracy of the scenario-based Value-at-Risk. When aggregating the
Value-at-Risks relating to the principal components, Eq. 25 does not make use of correlations since the scores of principal components are by definition uncorrelated. A natural generalization of Eq. 25 is to allow for correlations when aggregating the Value-at-Risks $\text{VaR}_k$:

$$\text{VaR}_{1-\alpha} (1 \text{ year}) \approx \sqrt{\sum_{k=1}^{\hat{K}} \sum_{l=1}^{\hat{K}} \rho_{k,l} \cdot \text{VaR}_k \cdot \text{VaR}_l - \left(\mathbb{E}[\mathbf{X}_1] - \mathbf{X}_0\right) \cdot \mathbf{S}}$$

(26)

with $\rho_{k,k} = 1$ for all $k = 1, ..., \hat{K}$. Campbell et al. (2002) suggest estimating the parameters $\rho_{k,l}$ implicitly, such that they imply an optimal fit between the aggregation based on the square-root formula and the exact Value-at-Risk of the portfolio. Campbell et al. (2008) highlight that those implied correlation parameters can be driven by fat distribution tails. In the aggregation of risk (sub-)modules in the Solvency-II-standard formula, the correlation parameters “are chosen in such a way as to achieve the best approximation of the 99.5% VaR for the overall (aggregated) capital requirement”, reflecting imperfections with this aggregation formula such as skewed distributions.\textsuperscript{14} Hence, even though the Pearson correlation between the principal component scores is zero, correlation parameters may be included in Eq. 26 to outweigh deficiencies resulting from skewed or fat-tailed distributions of the principal component scores and thereby to improve the accuracy of the approximation.

\textsuperscript{14}The verbatim quote is from EIOPA (2014b, p. 8).
Mittnik (2014) suggests identifying the correlation parameters that ensure an optimal fit of Eq. 26 simultaneously for various portfolios. Transferring this idea to our context means that the correlation parameters should minimize

\[ \sum_{i=1}^{N} \left( \hat{\text{VaR}}(i) - \text{VaR}(i) \right)^2 \]  

(27)

where \( \hat{\text{VaR}}(i) \) is the approximate Value-at-Risk for portfolio \( i \) according to the right-hand side of Eq. 26, \( \text{VaR}(i) \) is the Value-at-Risk according to the interest-rate model from section 2.2 and \( N \) is the number of portfolios. According to Mittnik (2014), the choice of portfolios should reflect practical considerations, such as asset allocation limits. If a regulator wants to use Eq. 26 in the context of a standard formula, the correlation parameters could be optimized with regard to the asset-liability portfolios of the firms subject to the regulatory jurisdiction.

3 Calibration of interest rate models

3.1 Data

The model calibration is based on data published by the ECB, which estimates the yield curve from AAA-rated Euro-area central government bonds with the Svensson model.\(^{15}\) The ECB has published the 6 parameters of the Svensson model, together with the corresponding interest rates for maturities 1, 5, 10, 20 and 30 years, for every trading day since 6 September 2004. We use these data on a daily basis from 6 September 2004 to 29 December 2017, giving us 3410 observations of the yield curve. Table 1 shows the

descriptive statistics for the daily changes in interest rates for maturities 1, 5, 10, 20 and 30 years.

Table 1: Descriptive statistics for daily changes in interest rates from 6.9.2004 to 29.12.2017 (in basis points).

<table>
<thead>
<tr>
<th>Maturity</th>
<th>Mean</th>
<th>Std Dev.</th>
<th>Min</th>
<th>Max</th>
<th>Skewness</th>
<th>Kurtosis</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.090</td>
<td>2.537</td>
<td>-26.368</td>
<td>19.440</td>
<td>-0.703</td>
<td>14.893</td>
</tr>
<tr>
<td>5</td>
<td>-0.107</td>
<td>3.810</td>
<td>-22.579</td>
<td>18.312</td>
<td>0.029</td>
<td>5.252</td>
</tr>
<tr>
<td>10</td>
<td>-0.108</td>
<td>3.849</td>
<td>-19.305</td>
<td>18.054</td>
<td>0.165</td>
<td>4.606</td>
</tr>
<tr>
<td>20</td>
<td>-0.107</td>
<td>4.213</td>
<td>-24.100</td>
<td>24.027</td>
<td>0.037</td>
<td>5.856</td>
</tr>
<tr>
<td>30</td>
<td>-0.107</td>
<td>4.851</td>
<td>-56.403</td>
<td>31.075</td>
<td>-0.298</td>
<td>11.360</td>
</tr>
</tbody>
</table>

3.2 Calibration

When calibrating the Log-DNS model, we set $r^{\text{min}} = -2\%$.\textsuperscript{16} For both models, the parameter $\lambda$ is estimated consistently with Caldeira et al. (2015, p. 74) by minimizing the expression

$$
\sum_{t=1}^{T} \sum_{i=1}^{m} (\hat{y}_t(\tau_i) - y_t(\tau_i))^2
$$

where $y_t(\tau_i) = r_t(\tau_i)$ in the case of the DNS model and $y_t(\tau_i) = \ln(r_t(\tau_i) - r^{\text{min}})$ in the case of the Log-DNS model, $\hat{y}_t = r_t - \epsilon_t$, the index $t$ runs from 6.9.2004 to 29.12.2017 and the index $i$ runs through the set of maturities $\{1, 5, 10, 20, 30\}$. Subsequently, the parameters $\beta_{1,t}, \beta_{2,t}, \beta_{3,t}$ are estimated per trading day by ordinary least square (OLS) regression.

In the observed time horizon from 2004 to 2017, interest rates have significantly decreased (cf. column “Mean” in Table 1). We remove this drift from the observed $\Delta f_t$-processes

\textsuperscript{16}For a discussion about lower bounds for interest rates, cf. Viñals et al. (2016), who state in the official blog of the International Monetary Fund that “Ballpark estimates by staff for the tipping point at which a move into cash would become worthwhile range from minus 75 basis points (bps) to minus 200 bps”.
of both models by deducting the mean of $\Delta f_t^{(i)}$ for each entry $i$. This helps to avoid the negative drift continuing in the simulated yield curves of the next year, which would drive interest rates in 1 year below the current level in expectation. For both models, the lag order $p$ of the VAR-process is chosen according to the criterion of Hannan and Quinn (1979) (HQ).\textsuperscript{17} Subsequently, the parameters $\mu$ and $\Gamma$ of the VAR model are estimated by OLS regression per equation. The parameters of the DCC model are estimated with R software using the rmgarch package from Ghalanos (2015).\textsuperscript{18} According to the Augmented Dickey-Fuller (ADF) test, the disturbances of the VAR-process in Eq. 8 are stationary. Finally, we estimate the parameters of the $\epsilon_t$-processes for $m = 5$ maturities 1, 5, 10, 20 and 30 years (in line with the maturities of the interest rates published by the ECB). The parameters of the AR(1)-processes for $\epsilon_t(\tau)$ are estimated by OLS regression and the variances of their disturbances are calculated by the unbiased variance estimator. According to the ADF test, these disturbances are stationary.

4 Backtesting

4.1 Backtesting interest rate models

We backtest the Value-at-Risks according to the DNS and Log-DNS models by comparing them with losses that would have been realized for the historical yield curve movements. Since the realized loss depends on the composition of the portfolio (which may exhibit

\textsuperscript{17}Shittu and Asemota (2009) demonstrate that this criterion outperforms the Akaike information criterion (AIC) and the Bayesian information criterion (BIC) for autoregressive processes and large samples.

\textsuperscript{18}The estimation works in two steps. In the first step, the parameters $\omega_i, \kappa_i, \lambda_i$ of Eq. 12 are determined by Maximum-Likelihood estimation. Using the predictions for $D_t$ according to Eq. 12, $z_t$ is determined based on Eq. 10. In the second step, $a$ and $b$ of Eq. 13 are estimated.
long or short exposures for each maturity), the analysis is carried out for 1,000 randomly generated asset-liability portfolios of hypothetical insurance companies. Each portfolio \( i \in \{1, \ldots, 1000\} \) consists of two cash inflows at the amount of 2 monetary units and two cash outflows at the amount of 1 monetary unit. The maturities of all cash flows were chosen based on independent random numbers; the calibration of their distribution takes the empirical results of Möhlmann (2017) into account. For each inflow, the maturity was chosen by a normal distribution with mean 10 and standard deviation 15; the realization was rounded to a whole number and bounded between 1 and 40. This leads to an average Macaulay duration of cash inflows of 10.0, which is close to German life insurers’ average asset duration of 9.9 (cf. Möhlmann, 2017, p. 10). The maturity of each cash outflow was chosen analogously, except for the normal distribution’s mean being 15 instead of 10. The average Macaulay duration of cash outflows is 14.9, which is close to German life insurers’ average liability duration of 14.7 (cf. Möhlmann, 2017). The duration gap of the hypothetical insurance companies is 5.8 on average (corresponding to 4.9 according to Möhlmann, 2017) with a standard deviation of 4.1 (corresponding to 3.9 according to Möhlmann, 2017).\(^{19}\)

The Solvency II capital requirement is the 99.5% Value-at-Risk with a 1-year holding period. Backtesting this value with historical data is impossible, since it would require interest rate data from at least 200 years. Instead, we conduct the backtesting for several combinations of holding periods \( h \) (in trading days) and confidence levels \( 1 - \alpha \) and check whether an increase in the holding period \( h \) systematically affects the accuracy of the Value-at-Risk estimate.

\(^{19}\)To be precise, Möhlmann’s (2017) estimate of the average duration gap at the amount of 3.9 is weighted by the size of insurance companies. The standard deviation of 3.9 refers to unweighted duration gaps, the average of which is 6.8.
The backtesting is conducted based on out-of-sample estimates. Hence, when calculating the Value-at-Risk as of day \( t \), we only use data between day 1 and day \( t \) to estimate

- the parameters \( \lambda, \beta_1, \beta_2, \beta_3 \) of Equations 5 and 7,
- the parameters \( \mu, \Gamma_k \) of Eq. 8,
- the parameters of the DCC-model (Eq. 12 and 13),
- the parameters of the AR(1) processes for the residuals \( \epsilon_t \) and
- the standard deviations of the disturbances of the AR(1) processes for \( \epsilon_t \).

To determine the Value-at-Risk as of day \( t \) over a holding period of \( h \) days, we then generate 10,000 simulations of the yield curves \( r_{t+1}, ..., r_{t+h} \), for both the Log-DNS and the DNS model. The Value-at-Risk for portfolio \( i \), \( \text{VaR}_{i(1-\alpha),h}(t) \), is compared with the historical loss that has occurred for portfolio \( i \) between times \( t \) and \( t+h \):

\[
\text{loss}_{t}^{(i)} = -\sum_{\tau=1}^{M} \left( e^{-\tau \cdot r_{t+h}(\tau)} - e^{-\tau \cdot r_{t}(\tau)} \right) \cdot CF_{\tau}^{(i)}
\]  

(28)

In order to avoid autocorrelation in the \( \text{loss}_{t}^{(i)} \)-processes, we conduct this comparison only beginning at every \( h \)-th day of the observed time period. In line with Caldeira et al. (2015, p. 73), the backtesting departs from day 500, such that at least 500 days can be used to calibrate the models. We can thereby observe \( n = \lfloor \frac{3410 - 500}{h} \rfloor \) pairwise disjunct time windows, each with a length of \( h \) days.

The percentage of days for which the historical loss exceeds the Value-at-Risk is called the hit rate:

\[
\text{hit rate} = \frac{1}{n} \sum_{t} \mathbb{1}_{\{\text{loss}_{t}^{(i)} > \text{VaR}_{i(1-\alpha),h}(t)\}}
\]  

(29)
For an accurate Value-at-Risk estimate for portfolio $i$, the hit rate should be close to $\alpha$, i.e. for about $\alpha \cdot n$ days, the historical loss should exceed the Value-at-Risk.

We conduct the analysis for $\alpha = 0.5\%$, $\alpha = 5\%$ and $\alpha = 10\%$ combined with holding periods of $h = 5$, $h = 15$, $h = 30$ and $h = 50$ days. These choices have been made in light of Solvency II regulations (for which $\alpha = 0.5\%$ in connection with a holding period of 1 year would be most relevant), calculation time (the smaller $h$, the larger the number of time windows for which all parameters need to be estimated) and sampling error. Sampling error impacts the hit rate more strongly the smaller $\alpha \cdot n$ is, since the number of expected hits becomes small. For instance, for $h = 5$, we can compare the Value-at-Risk and the historical loss for $\left\lfloor \frac{3410 - 500}{5} \right\rfloor = 582$ time windows. For $h = 50$, we obtain only 58 time windows, meaning that only the hit rate of the 90% Value-at-Risk remains relatively robust.

The aim of the subsequent analysis is to examine (1) whether a model provides a proper fit in the distribution tail in order to estimate the 99.5% Value-at-Risk, (2) whether a model becomes systematically more optimistic or more conservative when extending the holding period and (3) whether Value-at-Risk exceedances are clustered in some time periods or occur independently of each other.

To address the first two questions, Table 2 shows the averages and standard deviations of the hit rates across the 1,000 asset-liability portfolios. In addition, we have checked for each portfolio and each combination of $h$ and $\alpha$ whether the hit rate deviates significantly from the desired level according to Christoffersen’s (1998) tests for unconditional coverage. The first part of Table 3 shows the portion of portfolios for which the Value-at-Risk deviates significantly at a 1%, 5% and 10% level of significance.
For the shortest holding period $h = 5 \text{ days}$, the results in Table 2 suggest that the Log-DNS model provides—on average across all portfolios—suitable estimates for the Value-at-Risk at all three confidence levels. The results in Table 3 confirm that the Value-at-Risk of a large portion of portfolios are not significantly inaccurate. For instance, the hit rate of the 99.5% Value-at-Risk deviates significantly from 0.5% only for 6% of portfolios at a 1% level of significance and for 15% of portfolios at a 10% level of significance. The 95% and 90% Value-at-Risks are suitable even for larger portions of portfolios in this sense. In comparison with the DNS model, the accuracy of the Value-at-Risks provided by the Log-DNS model tends to be better rather than worse. Two conclusions can be drawn from these results: firstly, the Log-DNS model’s advantage of respecting a lower bound for interest rates does not come at a disadvantage in terms of the accuracy. Secondly, the Value-at-Risk according to the (Log)-DNS model is suitable not only for an equally-weighted asset portfolio, as demonstrated by Caldeira et al. (2015), but also for various asset-liability portfolios. In total, this suggests that both models meet expectations towards a regulatory standard formula of suitability for typical insurance companies.

Looking at the development of the average hit rates of the 90% and 95% Value-at-Risk when extending the holding period provides little evidence of a systematic change in the accuracy. Hence, both models appear to be suitable for longer holding periods as well.

Next, we investigate dependencies of Value-at-Risk exceedances over time. To this end, we employ the independence test of Christoffersen (1998) on the null hypothesis that a Value-at-Risk exceedance does not affect the probability of an exceedance in the subsequent time window. The second part of Table 3 shows the portion of portfolios for which the pattern of Value-at-Risk exceedances is significantly dependent over time.
Table 2: Averages and standard deviations of hit rates across 1,000 portfolios.

<table>
<thead>
<tr>
<th>Model</th>
<th>Holding period (in days)</th>
<th>(0.5%)</th>
<th>Std Dev.</th>
<th>(5%)</th>
<th>Std Dev.</th>
<th>(10%)</th>
<th>Std Dev.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Log-DNS</td>
<td>5</td>
<td>0.63%</td>
<td>0.40%</td>
<td>4.77%</td>
<td>0.68%</td>
<td>9.04%</td>
<td>0.72%</td>
</tr>
<tr>
<td></td>
<td>15</td>
<td>0.77%</td>
<td>0.58%</td>
<td>5.79%</td>
<td>0.93%</td>
<td>8.54%</td>
<td>0.79%</td>
</tr>
<tr>
<td></td>
<td>30</td>
<td>0.68%</td>
<td>0.69%</td>
<td>5.19%</td>
<td>1.08%</td>
<td>8.59%</td>
<td>1.50%</td>
</tr>
<tr>
<td></td>
<td>50</td>
<td>0.72%</td>
<td>0.85%</td>
<td>4.05%</td>
<td>1.50%</td>
<td>7.70%</td>
<td>2.29%</td>
</tr>
<tr>
<td>DNS</td>
<td>5</td>
<td>0.65%</td>
<td>0.37%</td>
<td>4.80%</td>
<td>0.57%</td>
<td>8.50%</td>
<td>0.50%</td>
</tr>
<tr>
<td></td>
<td>15</td>
<td>1.36%</td>
<td>0.73%</td>
<td>5.34%</td>
<td>0.78%</td>
<td>8.58%</td>
<td>1.14%</td>
</tr>
<tr>
<td></td>
<td>30</td>
<td>0.45%</td>
<td>0.54%</td>
<td>4.61%</td>
<td>1.10%</td>
<td>7.84%</td>
<td>2.22%</td>
</tr>
<tr>
<td></td>
<td>50</td>
<td>1.25%</td>
<td>0.77%</td>
<td>3.57%</td>
<td>1.16%</td>
<td>7.13%</td>
<td>1.57%</td>
</tr>
</tbody>
</table>

The 95% and 99.5% Value-at-Risks exhibit for both models, all considered holding periods and (almost) all portfolios no patterns of significantly clustered exceedances. For a relatively large portion of portfolios, exceedances of the 90% Value-at-Risk are dependent at a 10% level of significance when the holding period is 5 or 15 days. Since from a Solvency II perspective, combinations of high confidence levels and long holding periods are most relevant, time-dependent Value-at-Risk exceedances should not be a major issue for the intended application of the models.

Finally, we have applied Christoffersen’s (1998) test for conditional coverage, which combines the tests for accuracy (unconditional coverage) and for the independence of Value-at-Risk exceedances. The last part of Table 3 demonstrates that in all considerations, the suitability of the Value-at-Risk according to the Log-DNS model cannot be rejected for at least 89% of portfolios.

4.2 Backtesting the scenario-based Value-at-Risk

The starting point and benchmark for the scenario-based approach consists of simulations of the yield curves after 1 year. To this end, we calibrate the Log-DNS model as on 29 December 2017 using the complete time series of yield curve data. We then generate 30,000 simulations for the interest rates \(r_{\text{year end } 2018}(\tau, \omega)\) for maturities \(\tau \in \{1, 5, 10, 20, 30\}\). We
Table 3: Portion of portfolios with p-value of Christoffersen’s Exceedance tests below 1%, 5%, and 10%.

<table>
<thead>
<tr>
<th>Test</th>
<th>Model</th>
<th>Holding period</th>
<th>99.5% p-value below</th>
<th>95% p-value below</th>
<th>90% p-value below</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>1%</td>
<td>5%</td>
<td>10%</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>1%</td>
<td>5%</td>
<td>10%</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>1%</td>
<td>5%</td>
<td>10%</td>
</tr>
<tr>
<td>Unconditional</td>
<td>Log-DNS</td>
<td>5</td>
<td>6%</td>
<td>15%</td>
<td>15%</td>
</tr>
<tr>
<td></td>
<td></td>
<td>15</td>
<td>1%</td>
<td>8%</td>
<td>21%</td>
</tr>
<tr>
<td></td>
<td></td>
<td>30</td>
<td>0%</td>
<td>0%</td>
<td>0%</td>
</tr>
<tr>
<td></td>
<td></td>
<td>50</td>
<td>0%</td>
<td>0%</td>
<td>0%</td>
</tr>
<tr>
<td>coverage</td>
<td>DNS</td>
<td>5</td>
<td>3%</td>
<td>16%</td>
<td>16%</td>
</tr>
<tr>
<td></td>
<td></td>
<td>15</td>
<td>12%</td>
<td>28%</td>
<td>49%</td>
</tr>
<tr>
<td></td>
<td></td>
<td>30</td>
<td>0%</td>
<td>0%</td>
<td>0%</td>
</tr>
<tr>
<td></td>
<td></td>
<td>50</td>
<td>0%</td>
<td>0%</td>
<td>0%</td>
</tr>
<tr>
<td>Independence</td>
<td>Log-DNS</td>
<td>5</td>
<td>0%</td>
<td>0%</td>
<td>0%</td>
</tr>
<tr>
<td></td>
<td></td>
<td>15</td>
<td>0%</td>
<td>0%</td>
<td>0%</td>
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<tr>
<td></td>
<td></td>
<td>30</td>
<td>0%</td>
<td>0%</td>
<td>0%</td>
</tr>
<tr>
<td></td>
<td></td>
<td>50</td>
<td>0%</td>
<td>0%</td>
<td>0%</td>
</tr>
<tr>
<td>coverage</td>
<td>DNS</td>
<td>5</td>
<td>0%</td>
<td>0%</td>
<td>0%</td>
</tr>
<tr>
<td></td>
<td></td>
<td>15</td>
<td>0%</td>
<td>0%</td>
<td>0%</td>
</tr>
<tr>
<td></td>
<td></td>
<td>30</td>
<td>0%</td>
<td>0%</td>
<td>0%</td>
</tr>
<tr>
<td></td>
<td></td>
<td>50</td>
<td>0%</td>
<td>0%</td>
<td>0%</td>
</tr>
<tr>
<td>Conditional</td>
<td>Log-DNS</td>
<td>5</td>
<td>3%</td>
<td>8%</td>
<td>11%</td>
</tr>
<tr>
<td>coverage</td>
<td>DNS</td>
<td>5</td>
<td>1%</td>
<td>4%</td>
<td>3%</td>
</tr>
</tbody>
</table>

set the time horizon to 254 days, which is the number of days with observable interest data in 2017.

To set up the scenarios, we transform the discount factors according to the simulated interest rates $r_{\text{year end } 2018}(\tau, \omega)$ with $\tau \in \{1, 5, 10, 20, 30\}$ into principal components (hence, $K = 5$ in Eq. 17). We then use Eq. 23 and 24 to elicit two stressed interest rates for each of the five modeled maturities and each principal component. Each set of five stressed interest rates is then extrapolated to a complete stressed yield curve by fitting the Svensson parameters.$^{20}$ Finally, the Value-at-Risk is calculated according to Eq. 25.

---

$^{20}$For simplicity, $\lambda_{1,t}$ and $\lambda_{2,t}$ are taken from 29.12.2017. Then $\beta_{1,t}, \ldots, \beta_{4,t}$ are fitted by OLS.
Table 4 provides the stressed interest rates corresponding to the 99.5% Value-at-Risk over a 1-year holding period for maturities 1, 5, 10, 20 and 30 years. The stressed interest rates are presented in terms of the absolute changes to the interest rates on 29 December 2017. Regarding the first principal component (PC1), the two stress scenarios A and B are essentially an upward and downward shift of interest rates. The second principal component (PC2) governs the steepness of the yield curve by changing long-term interest rates in a different direction than short and middle-term rates. The third principal component (PC3) changes the yield curve at the short end and can, in connection with PC1 and PC2, govern the curvature.

At first glance, the yield curve scenarios in Table 4 may appear conservative. For instance, scenario A of PC2 would lead to a clearly inverted yield curve. However, one must recall firstly that the scenarios reflect how the yield curve can change over one year in very rare cases (once in 200 years), and secondly that the scenario-based Value-at-Risk in total is not only based on the strictest scenario, but allows for diversification effects between the scenarios of different principal components, which reduces the Value-at-Risk.

Table 4: Interest rate stress scenarios (in absolute changes to yield curve on 29 December 2017).

<table>
<thead>
<tr>
<th>Maturity</th>
<th>1 year</th>
<th>5 years</th>
<th>10 years</th>
<th>20 years</th>
<th>30 years</th>
</tr>
</thead>
<tbody>
<tr>
<td>PC1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>A</td>
<td>-1.4%</td>
<td>-1.8%</td>
<td>-2.1%</td>
<td>-1.8%</td>
<td>-1.6%</td>
</tr>
<tr>
<td>B</td>
<td>2.1%</td>
<td>2.7%</td>
<td>3.5%</td>
<td>3.9%</td>
<td>4.8%</td>
</tr>
<tr>
<td>PC2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>A</td>
<td>3.3%</td>
<td>2.1%</td>
<td>1.2%</td>
<td>0.1%</td>
<td>-0.3%</td>
</tr>
<tr>
<td>B</td>
<td>-0.9%</td>
<td>-0.5%</td>
<td>-0.2%</td>
<td>0.1%</td>
<td>0.2%</td>
</tr>
<tr>
<td>PC3</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>A</td>
<td>1.2%</td>
<td>0.5%</td>
<td>0.0%</td>
<td>0.1%</td>
<td>0.1%</td>
</tr>
<tr>
<td>B</td>
<td>-1.4%</td>
<td>-0.4%</td>
<td>0.3%</td>
<td>0.2%</td>
<td>0.0%</td>
</tr>
<tr>
<td>PC4</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>A</td>
<td>0.8%</td>
<td>0.1%</td>
<td>0.1%</td>
<td>0.1%</td>
<td>0.1%</td>
</tr>
<tr>
<td>B</td>
<td>-0.4%</td>
<td>0.2%</td>
<td>0.1%</td>
<td>0.1%</td>
<td>0.1%</td>
</tr>
<tr>
<td>PC5</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>A</td>
<td>0.2%</td>
<td>0.1%</td>
<td>0.1%</td>
<td>0.1%</td>
<td>0.1%</td>
</tr>
<tr>
<td>B</td>
<td>0.2%</td>
<td>0.2%</td>
<td>0.1%</td>
<td>0.1%</td>
<td>0.1%</td>
</tr>
</tbody>
</table>
We backtest the scenario-based 99.5% Value-at-Risk over a 1 year holding period as at year end 2017 by comparing it to the corresponding “exact” Value-at-Risk which is based on the entire simulation. The underlying portfolios are the 1,000 asset-liability portfolios from section 4.1. Figure 1, part A, depicts the simulation-based Value-at-Risk (x axis) and the scenario-based Value-at-Risk (y axis) for the 1,000 portfolios. The y coordinates of the black points have been calculated based on the four scenarios of PC1 and PC2. The y coordinates of the gray points have been calculated with the two scenarios of PC1 only. When the coordinates of a portfolio lie on the bisector, the scenario-based Value-at-Risk coincides with the simulation-based Value-at-Risk.

Figure 1: 99.5% Value-at-Risk over a 1-year holding period as at year end 2017 of 1,000 portfolios according to the entire simulation (x-axis) and scenarios (y-axis).

The results shown in Figure 1 indicate that the scenario-based Value-at-Risk clearly understates the risk when it is determined based only on the two scenarios of PC1. For some portfolios, the scenario-based method would result in a Value-at-Risk close to zero, whereas the Value-at-Risk based on the entire simulation suggests a substantial risk. Those scenarios might represent insurers who are immunized by duration match-
ing against yield curve shifts, but not against changes in the yield curve’s steepness or curvature. Calculating the Value-at-Risk based on the four scenarios of PC1 and PC2 improves the accordance of the scenario-based with the simulation-based Value-at-Risk. To measure the degree of this accordance, we determine the root mean squared error (RMSE), which is calculated as

$$\sqrt{\frac{1}{1000} \sum_{i=1}^{1000} (\hat{\text{VaR}}(i) - \text{VaR}(i))^2}$$

(30)

where \(\text{VaR}(i)\) denotes the simulation-based Value-at-Risk and \(\hat{\text{VaR}}(i)\) denotes the scenario-based Value-at-Risk of portfolio \(i\).

By using the scenarios of PC2 in addition to those of PC1, the RMSE of the approximation reduces by about 55% from 0.218 to 0.098. Using the scenarios of PC3 in addition to those of PC1 and PC2 can further reduce the RMSE by only 2.8% from 0.098 to 0.096. Adding the scenarios of further principal components has hardly any impact on the accuracy of the approximation.

In order to improve the accuracy, we now implement correlation parameters in the scenario-based Value-at-Risk (cf. section 2.4). Using the first \(\hat{K} = 2\) principal components, the scenario-based Value-at-Risk is calculated as

$$\hat{\text{VaR}}(i) = \sqrt{[\text{VaR}_1(i)]^2 + 2 \cdot \rho_{1,2}^{(i)} \cdot \text{VaR}_1(i) \cdot \text{VaR}_2(i) + [\text{VaR}_2(i)]^2 - (\mathbb{E}[X_1] - X_0)' \cdot S}$$

where \(\text{VaR}_k(i)\) is the Value-at-Risk of portfolio \(i\) relating to the \(k\)th principal component, \(\rho_{1,2}^{(i)} = \rho_{1,2}^{\text{down}}\) if the downward scenario is relevant to determine \(\text{VaR}_1(i)\) and \(\rho_{1,2}^{(i)} = \rho_{1,2}^{\text{up}}\)
if the upward scenario is relevant to determine \( \text{VaR}_1(i) \).\(^{21}\) In the objective function (cf. line 27), we use the 1,000 portfolios from the backtesting exercises, which are assumed to reflect the regulated insurance companies in the market. The optimal correlation parameters are \( \rho_{1,2}^{\text{down}} = -0.441 \) and \( \rho_{1,2}^{\text{up}} = 0.607 \), which reduce the RMSE from 0.098 (scenario-based Value-at-Risk using PC1 and PC2) by 72% to 0.027. Part B of Figure 1 shows that the scenario-based Value-at-Risk provides a good fit for most portfolios. For portfolios with a relatively small risk as well as for those with a relatively high risk, the scenario-based Value-at-Risk is slightly too conservative, which appears to be in line with the spirit of a regulatory standard formula.

In total, the first two principal components in connection with two correlation parameters \( \rho_{1,2}^{\text{down}} \) and \( \rho_{1,2}^{\text{up}} \) provide a good approximation of the simulation-based Value-at-Risk. To verify the robustness of this result, Table 5 provides the RMSE’s when redoing the calculations based on data from 2004 until the end of year \( x \), with \( x \in \{2013, \ldots, 2016\} \).

Using PC2 in addition to PC1 reduces the RMSE by between 4% and 63%, where the reduction by only 4% occurred in a situation in which the RMSE for the calculation based on PC1 is already at a relatively low level. Using additional principal components has either no substantial effect or even increases the RMSE. Incorporating correlation coefficients which are optimal according to the criterion in section 2.4 reduces the RMSE by between 29% and 72% compared with the RMSE for the calculation based on PC1 and PC2. Finally, in order to rule out that the correlation parameters are only optimal with respect to the RMSE,\(^{22}\) we have also calculated all figures in Table 5 based on the

\(^{21}\)Differentiating the correlation parameter based on the downward and upward scenario is analogous to the Solvency II standard formula, where the correlation parameter between the the interest rate risk submodule and some other market risk submodules is set in dependence upon which interest rate scenario creates the higher loss.

\(^{22}\)According to the objective for the optimization of the correlation coefficients in Eq. 27, the parameters also minimize the RMSE.
Mean Absolute Error (MAE) across all portfolios. In comparison with the MAE for the Value-at-Risks based on PC1 and PC2, the incorporation of the correlation parameters reduces the MAE by between 27% (data 2004-2015) and 73% (data 2004-2016).

Table 5: Root mean squared error (RMSE) of scenario-based Value-at-Risk in comparison to simulation-based Value-at-Risk across 1,000 portfolios. Numbers in brackets show the relative change in RMSE by including an additional principal component (scenario-based Value-at-Risk with correlations: numbers in brackets show the isolated relative change in RMSE by including two correlation parameters, $\rho_{1,2}^{\text{down}}$ and $\rho_{1,2}^{\text{up}}$, and using PC1 and PC2).

<table>
<thead>
<tr>
<th>Data from 2004 to 20XX</th>
<th>Without correlations, using PC1 to PC5</th>
<th>With correlations PC1-PC2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>PC1</td>
<td>PC2</td>
</tr>
<tr>
<td>2013</td>
<td>0.053</td>
<td>0.020 (-63%)</td>
</tr>
<tr>
<td>2014</td>
<td>0.055</td>
<td>0.053 (-4%)</td>
</tr>
<tr>
<td>2015</td>
<td>0.223</td>
<td>0.107 (-52%)</td>
</tr>
<tr>
<td>2016</td>
<td>0.340</td>
<td>0.181 (-47%)</td>
</tr>
<tr>
<td>2017</td>
<td>0.218</td>
<td>0.098 (-55%)</td>
</tr>
</tbody>
</table>

5 Interest-rate sensitive cash flows

The Value-at-Risk approximation based on the principal component analysis in section 2.3 builds on the assumption that the portfolio losses are linear in the discount factors, meaning that the cash inflows and outflows remain constant when interest rates change. This assumption is largely fulfilled in non-life insurance and in simple life insurance contracts or pension plans which only provide fixed future benefits (as long as the insurer only makes limited use of financial derivatives linked to interest rate movements). In general, however, life insurers’ cash flows may vary in interest rates which may occur in particular through two different channels.

Firstly, the benefits to policyholders may include guaranteed and non-guaranteed components. In Germany, for instance, guarantees are typically embedded in a cliquet-style, meaning that the insurer grants a guaranteed return on the policyholder’s account each year. In addition, the insurer needs to credit at least 90% of book value investment re-
turns to policyholders. Modeling the impact of interest rates on benefits is not trivial and requires various assumptions, for instance, about the insurer’s hidden reserves and the investment policy.\textsuperscript{23} Secondly, life insurance contracts often include an option to surrender the contract before maturity. The level of interest rates may influence policyholders’ decision to surrender, as it alters the contract’s attractiveness in comparison with investment alternatives. Modeling interest rates’ impact on surrender decisions is also quite complex and will require assumptions about policyholders’ rationality (cf., e.g., Li and Szimayer, 2014; Cheng and Li, 2018). Analogously to surrender effects in the back book, interest rates may impact an insurer’s new business.

In the sense of section 2.3, interest-rate sensitive cash flows can increase the gap between the scenario-based and the (true) simulation-based calculations of the Value-at-Risk. The size of this gap will be driven by various parameters of the insurer’s contract portfolio and the business strategy. Due to the complexity of extending the model in this direction, we discuss only briefly and on a qualitative level how the accuracy of the Value-at-Risk approximation could be improved. The two most nearby ideas for this purpose are to modify the yield curve scenarios or the correlation parameters. It appears advantageous to alter the correlation parameters only, but not the scenarios. If the scenarios were altered, they would not reflect pure historical interest rate movements anymore and it would be much more difficult to interpret the scenario outcomes (in particular for cash flows on the asset side). To derive a suitable practical approach, one would need to define homogenous groups of insurance companies and obtain optimal correlation parameters for those groups.

\textsuperscript{23}For possible starting points see, for instance, Kling et al. (2007); Berdin and Gründl (2015).
6 Conclusion

When determining a method for the definition of capital requirements for financial institutions, regulators traditionally need to deal with a trade-off: on the one hand, they could force the companies to develop and use an internal model, which measures risk comprehensively based on Monte-Carlo simulations. However, this requires complex processes on the firm side to implement and maintain the models as well as on the regulator side to supervise them. On the other hand, capital requirements can be formulated by a pragmatic standard formula, which, however, often involves mismeasurements that can create severe disincentives for risk management decisions. This paper suggests a procedure for translating Monte-Carlo simulations for yield curves into a small number of scenarios, as well as a method for aggregating them towards a Value-at-Risk figure. By building on a modified version of the dynamic Nelson-Siegel model, the scenarios respect a lower bound for interest rates.

The procedure offers regulators a way to define pragmatic and at the same time risk-adequate capital requirements. The regulator would need to regularly calibrate the stochastic model, derive the scenarios and optimize the correlation parameters for the aggregation of the scenario outcomes with respect to the asset-liability portfolios of companies that employ the standard formula. The companies only need to recalculate their capital based on a small number of scenarios and use the aggregation formula to determine the Value-at-Risk. Moreover, the proposed methodology can be useful for companies that use an internal model. The translation of simulated yield curves into scenarios can be helpful to interpret, communicate and thereby better use internal model results, which is traditionally challenging due to the large number of simulation paths. In this con-
text, the correlation parameters for the aggregation can be optimized with respect to asset-liability portfolios resulting from business decisions which shall be supported by the internal model.
References


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