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Sensitivity-implied tail-correlation matrices*

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Abstract

Tail-correlation matrices are an important tool for aggregating risk measurements across risk categories, asset classes and/or business segments. This paper demonstrates that the classical concepts, such as VaR-implied correlations, can lead to substantial biases of the aggregate risk measurement's sensitivities with respect to risk exposures. Due to these biases, decision-makers receive an odd view of the effects of portfolio changes and may be unable to identify the optimal portfolio from a risk-return perspective. To overcome these issues, we introduce the "sensitivity-implied tail-correlation matrix", whose entries are calculated as second-order partial derivatives of the squared risk measurement. The proposed tail-correlation matrix allows for a simple deterministic risk aggregation approach which reasonably approximates the "true" aggregate risk measurement according to the complete multivariate risk distribution. Specifically, a risk measurement based on the proposed matrix correctly reflects the risk of the calibration portfolio as well as all first and second-order sensitivities. Numerical examples demonstrate that our approach is a better basis for portfolio optimization than the VaR-implied tail-correlation matrix, especially if the calibration portfolio (or current portfolio) deviates from the optimal portfolio.

JEL classification: G11, G22, G28, G32.

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1 Introduction

Tail-correlation matrices offer an approach to aggregate risks in a simple deterministic manner. The correlation-based risk aggregation is employed in various contexts, including the calculation of regulatory capital requirements or a firm's economic capital. Regarding an n-risks-portfolio, the approach starts from n univariate risk measurements, which are collected in a vector $x \in \mathbb{R}^n$. Then, an $n \times n$-matrix R of correlation parameters is used to calculate the aggregate risk measurement as

$$\sqrt{x^{\mathrm{T}} R x} \tag{1}$$

The approach in line (1) is employed in the Solvency II standard formula, which is used to determine the regulatory capital requirement for most insurance companies in the European Union (EU). The Solvency II standard formula is calibrated to reflect the 99.5% Value-at-Risk of an insurer's asset-liability portfolio over a one-year time horizon. Apart from the EU, the approach (1) is used in insurance regulation in the United States ("Risk-Based Capital"), China ("C-ROSS") and the International Capital Standard. In the banking industry, the approach is referred to as the variance-covariance approach and is popular in banks' internal risk assessments (cf. Mathur, 2015, pp. 272-274; Li et al., 2015). Moreover, the approach can be used for investment portfolio optimization (cf. Mittnik, 2014). Structurally, the calculation of the portfolio risk using (1) mimics the calculation of the standard deviation of portfolio risk. Hence, portfolio selection problems in connection with the risk measurement in (1) can be studied analogously to those of the mean-variance framework of Markowitz (1952).[1]

[1] Apart from investment portfolio optimization, the mean-variance framework has been employed in an insurance context. For example, Eckert and Gatzert (2018) identify an insurer's optimal risk-return

Using correlation parameters derived from the covariance matrix, approach (1) can guarantee an exact aggregation of risk measurements only if risks follow a multivariate elliptical risk distribution (cf. McNeil et al., 2015, pp. 295 ff.). If risks exhibit heavy tails or non-linear dependencies,[2] the aggregate risk measurement based on (1) can substantially differ from the "true" result in accordance with the complete multivariate risk distribution (Pfeifer and Strassburger, 2008; Li et al., 2015).[3] To eliminate this bias and in connection with the risk measure Value-at-Risk (VaR), so-called VaR-implied tail-correlations have been proposed (Campbell et al., 2002; Mittnik, 2014). According to EIOPA (2014, p. 9), the risk aggregation in the Solvency II standard formula has been calibrated based on VaR-implied tail-correlations.

Chen et al. (2019) empirically study the sensitivities of the correlation-based risk aggregation approach with regard to the regulatory Risk-Based Capital (RBC) for US insurance companies, which is referred to as the "square-root formula" in this context.[4] The authors find that the insurers' optimal investment policy is driven by marginal capital requirements, i.e. by sensitivities of the aggregate capital requirement with respect to the size of univariate risks. Moreover, the authors demonstrate that the square-root formula has understated the marginal capital requirement of fixed-income investments and has thereby

combination against the background of policyholders' willingness to pay depending on the insurer's solvency level. Braun et al. (2017) investigate insurers' asset allocations in a mean-variance framework when they face a regulatory capital requirement determined by the Solvency II standard formula. Braun et al. (2017) find that the standard formula tends to promote inefficient portfolios over efficient ones.

[2]Empirical evidence indicates that correlations between asset returns are higher during periods of (stressful) downside moves, cf. for example Longin and Solnik (2001), Campbell et al. (2002) and Ang and Chen (2002). In addition, risk types such as operational risks or non-life insurance risks follow more heavily tailed distribution, see for example Bernard et al. (2018).

[3]In addition, Breuer et al. (2010) find that the summation of the regulatory capitals for market risks and credit risks—which can be viewed as a special case of (1) with R including only ones—is not necessarily conservative given that many financial positions can be affected by both types of risks.

[4]More precisely, Chen et al. (2019) consider property and casualty insurance companies. For those insurers, the RBC includes six risk categories, such as stock risk, underwriting risk and reserving risk. Five of the risk categories are assumed to be uncorrelated and one of them (affiliated investments) is added up on the result of the square-root formula.

incentivized insurers to increase those investments. The insurers' overall risks have thus increased.

Our paper elaborates on the observations of Chen et al. (2019) in a stylized set-up. We demonstrate that the sensitivities of the approach (1) can be substantially biased if R is a VaR-implied tail-correlation matrix, even if the calibration of R is conducted based on the complete multivariate risk distribution.

To make the correlation-based risk aggregation approach a suitable basis for portfolio management decisions, we propose taking a different view of matrix R. We show that for elliptical distributions, the entries of R globally coincide with the second-order partial derivatives of the squared aggregate risk measurement with respect to changes in the risk measurements of the univariate risks. For general distributions, these second-order partial derivatives uniquely define a symmetric matrix. We show that the approach (1) in connection with this *"sensitivity-implied tail-correlation matrix"* is a useful local approximation of the true aggregate risk measurement: for the calibration portfolio, it yields the aggregate risk and all first and second-order sensitivities with respect to risk exposures in line with the respective results based on the true risk distribution. If the distribution is not elliptical, the diagonal elements of the sensitivity-implied tail-correlation matrix may deviate from one.

To analyze the implications of the calibration of the matrix R for portfolio optimization and business steering, we consider an example of a multiline-insurance company whose objective is the maximization of Economic Value Added (EVA) in connection with the risk measure 99.5% VaR. As a basis of comparison, we identify the "true" EVA-optimal strategy by calculating the risk measure based on the true multivariate risk distribution. Afterwards, we derive the EVA-maximizing strategy if the correlation-based risk aggregation approach is used in connection with either a VaR-implied tail-correlation matrix or our proposed sensitivity-implied tail-correlation matrix. We find that the use of a traditional VaR-implied tail-correlation matrix induces a strategy which achieves a reduced EVA and goes along with a lower safety level than the true EVA-optimal strategy. In combination with the sensitivity-implied tail-correlation matrix, those distortions are very small, even if the calibration portfolio of the sensitivity-implied tail-correlation matrix clearly differs from the true EVA-optimal portfolio.

The remainder of this paper is structured as follows. Subsequent to defining the set-up in section 2, section 3 provides an overview of the classical VaR-implied tail-correlation matrices and introduces the sensitivity-implied tail-correlation matrix. Section 4 provides numerical examples including an analysis in terms of EVA optimization. Section 5 compares the sensitivity-implied tail-correlation matrix with VaR-implied tail-correlation matrices in terms of their properties and their calibration. Section 6 concludes and outlines possible areas of application of the sensitivity-implied tail-correlation matrix.

2 Set-up

Suppose the profit and loss of a portfolio over a specified period of time is given by

$$\sum_{i=1}^{n} u_i \cdot X_i, \tag{2}$$

where $n \in \mathbb{N}$ denotes the number of relevant risks, $(X_1, ..., X_n)^{\mathrm{T}}$ is a random vector with $\mathbb{E}(X_i) < \infty$ for all $i \in \{1, ..., n\}$ and $u_i \in \mathbb{R}$ reflects the exposure to risk i. Going forward, we assume that the multivariate distribution of $X = (X_1, ..., X_n)^{\mathrm{T}}$ is fixed and that the variable u fully specifies the portfolio.[5] Moreover, in line with Tasche (2008), we assume that the X_i are scaled such that the coordinates $u = \mathbb{1}_n = (1, \dots, 1)^{\mathrm{T}}$ reflect the current portfolio. Following the notation in Tasche (2008), the function $f_{\varrho,X}$ measures the "true" aggregate risk of portfolio u,

$$
\begin{aligned}
& f_{\varrho,X} : \mathbb{R}^n \to \mathbb{R}, \\
u = (u_1, \dots, u_n)^{\mathrm{T}} \mapsto f_{\varrho,X}(u) &= \varrho\left(\sum_{i=1}^{n} u_i \cdot X_i\right) - \mathbb{E}\left(\sum_{i=1}^{n} u_i \cdot X_i\right),
\end{aligned} \tag{3}
$$

with ϱ being a risk measure which is homogeneous of degree one, law invariant and translation invariant. The vector $x \in \mathbb{R}^n$ contains the univariate risk measurements in accordance with ϱ. The entries of x are defined as

$$
x_i = \varrho(X_i) - \mathbb{E}(X_i), \quad i = 1, ..., n \tag{4}
$$

The function g measures the aggregate risk of portfolio u based on the risk aggregation approach (1), depending on the risk measurement x and the matrix $R = (\varrho_{ij})_{i,j=1}^{n} \in \mathbb{R}^{n \times n}$:

$$
g : U \to \mathbb{R}, \quad u = (u_1, \dots, u_n)^{\mathrm{T}} \mapsto g(u) = \sqrt{(u \circ x)^{\mathrm{T}} R \, (u \circ x)}, \tag{5}
$$

where $\circ$ denotes the Hadamard product, i.e. $u \circ x = (u_1 x_1, \dots, u_n x_n)^{\mathrm{T}} \in \mathbb{R}^n$, and $U \subset \mathbb{R}^n$ is defined such that $(u \circ x)^{\mathrm{T}} R \, (u \circ x)$ is positive for all $u \in U$. We assume $\mathbb{1}_n \in U$.

[5] Assuming a linear relationship between the portfolio return and the exposure vector u is popular in the related literature, cf. for example Zanjani (2002); Tasche (2008); Hong and Liu (2009); Buch et al. (2011); Mittnik (2014). An approach to generalize the relationship is presented by Boonen et al. (2017).

3 Tail-correlation matrices

3.1 VaR-implied tail-correlation matrix

The VaR-implied tail-correlation matrix is the most relevant traditional method for calibrating the matrix R.[6] It is a symmetric matrix with ones on its diagonal. Hence, for $n = 2$ risks and risk measurements $x_1 > 0, x_2 > 0$, there is only one free correlation parameter $\varrho_{1,2}$ which is to be set such that the approach (1) correctly determines the aggregate risk of the current portfolio (cf. Campbell et al., 2002, p. 89):

$$
\begin{aligned}
g\left(\mathbb{1}_2\right) &= f_{\varrho,X}\left(\mathbb{1}_2\right) \\
\Rightarrow x_1^2 + 2\varrho_{1,2}x_1x_2 + x_2^2 &= \left(f_{\varrho,X}\left(\mathbb{1}_2\right)\right)^2 \\
\Rightarrow \varrho_{1,2} &= \frac{\left(f_{\varrho,X}\left(\mathbb{1}_2\right)\right)^2 - x_1^2 - x_2^2}{2x_1x_2}
\end{aligned}
\tag{6}
$$

For $n \geq 3$ risks, Mittnik (2014, p. 70 f.) proposes defining the matrix R based on a set of ℓ calibration portfolios $w_1, ..., w_\ell \in \mathbb{R}^n$. Assuming that R is symmetric with ones on the diagonal, Mittnik proposes identifying the $n \cdot (n-1)/2$ free correlation parameters in R such that the mean squared error (MSE) of the function g^2 with regard to $f^2_{\varrho,X}$ across the ℓ calibration portfolios $w_1, ..., w_\ell \in \mathbb{R}^n$,

$$
\frac{1}{\ell}\sum_{k=1}^{\ell}\left(f^2_{\varrho,X}\left(w_k\right) - g^2\left(w_k\right)\right)^2, \tag{7}
$$

[6] The definitions of the VaR-implied tail-correlation matrix that we refer to in this section are compatible with other risk measures apart from Value-at-Risk, as long as they meet the conditions specified in section 2. Nevertheless, we continue to use the term "VaR-implied tail-correlation matrix" going forward.

is minimized.[7] Mittnik (2014) distinguishes between an "exact" identification if the number of portfolios coincides with the number of correlation parameters to be determined, $\ell = n(n-1)/2$, and an "overidentified" calibration if the number of portfolios exceeds the number of correlation parameters, i.e. $\ell > n(n-1)/2$. The term $f^2_{\varrho,X}(w_k) - g^2(w_k)$ in (7) is linear in the correlation parameters ϱ_{ij}. Hence, with an exact calibration of R, the risk assessments of $g(u)$ and $f_{\varrho,X}(u)$ coincide for all calibration portfolios.

Devineau and Loisel (2009, section 5) define R as the "minimal standard R" which solves the optimization problem

$$\begin{aligned} \|R\| &\to \min \\ \text{subject to } f(\mathbb{1}_n) &= g(\mathbb{1}_n), \end{aligned} \tag{8}$$

with the norm $\|\cdot\|$ being defined as $\|D\| = \sqrt{\text{trace}(D \cdot D^{\mathrm{T}})}$. Devineau and Loisel (2009) employ the approach only for $n = 2$ risks. For this case, the authors state that the problem in (8) is solved by (6).[8] For $n \geq 3$ risks, the matrix R calibrated according to (8) does not reflect which of the risks are more or less interdependent, since the calibration is only based on the diversified risk measurement and the stand-alone risk measurements. Our numerical examples in section 4 will illustrate this issue. More generally, our examples will demonstrate that the function $g(u)$ in connection with a VaR-implied tail-correlation matrix can misstate the slope and curvature of $f_{\varrho,X}(u)$, and that these misstatements can induce severe distortions in portfolio optimization.

[7] The objective function in line (7) corresponds to the least-squares estimator which Mittnik (2014, p. 71) uses in Eq. (11).

[8] It thereby becomes clear that Devineau and Loisel (2009) restrict R to have ones on the diagonal.

3.2 Sensitivity-implied tail-correlation matrix

To overcome the issues of the VaR-implied tail-correlation matrix, this section proposes a new type of tail-correlation matrix for the case that the function $f_{\varrho,X}(u)$ is twice continuously differentiable.[9] Proposition 1 states that $g(u)$ locally approximates $f_{\varrho,X}(u)$ if R is chosen based on second-order sensitivities of $f^2_{\varrho,X}(u)$.

Proposition 1. *Let ϱ be a risk measure which is homogeneous of degree one, law invariant and translation invariant and let $X = (X_1, ..., X_n)^{\mathrm{T}}$ be a risk vector. Let $M \subseteq \mathbb{R}^n$ be open with $\mathbb{1}_n \in M$. Assume that the function $f_{\varrho,X}(u)$, as defined in formula (3), is twice continuously differentiable on M and $f_{\varrho,X}(\mathbb{1}_n) > 0$. Let $x \in \mathbb{R}^n$ be defined as in (4) and assume that $x_k > 0$ for all $k = 1, \ldots, n$. Then, the matrix $R = (\varrho_{k\ell})^n_{k,\ell=1}$ defined by*

$$\varrho_{k\ell} = \frac{1}{2x_k x_\ell} \frac{\partial^2}{\partial u_k \partial u_\ell} f^2_{\varrho,X}(\mathbb{1}_n) \tag{9}$$

is symmetric. In combination with this matrix R, the function g, as defined in (5), has the following properties:

$$g(\mathbb{1}_n) = f_{\varrho,X}(\mathbb{1}_n), \tag{10}$$

$$\frac{\partial}{\partial u_\ell} g(\mathbb{1}_n) = \frac{\partial}{\partial u_\ell} f_{\varrho,X}(\mathbb{1}_n), \quad 1 \leq \ell \leq n, \tag{11}$$

$$\frac{\partial^2}{\partial u_k \partial u_\ell} g(\mathbb{1}_n) = \frac{\partial^2}{\partial u_k \partial u_\ell} f_{\varrho,X}(\mathbb{1}_n), \quad 1 \leq k, \ell \leq n. \tag{12}$$

We call the matrix R whose entries are defined in (9) the "sensitivity-implied tail-correlation matrix".[10] In connection with this matrix R, $g(u)$ approximates $f_{\varrho,X}(u)$ in

[9] Differentiability of $f_{\varrho,X}$ is commonly assumed in the context of Euler capital allocation, cf., for example, Tasche (2008). Moreover, using second-order derivatives can be essential in the context of portfolio optimization, cf. Buch et al. (2011).

[10] Similarly to the term "VaR-implied tail-correlation matrix", we call the matrix a *tail*-correlation matrix in order to point out that it does not have the properties of an ordinary correlation matrix.

the sense that it correctly determines the aggregate risk of the current portfolio, the sensitivities of the aggregate risk with respect to the exposures of all risks (starting at the current portfolio) as well as the corresponding second-order sensitivities with respect to all combinations of risks.

Note that $g(u)$ can be a more useful approximation of $f_{\varrho,X}(u)$ than a Taylor polynomial: $g(u)$ is, like $f_{\varrho,X}(u)$, homogeneous of degree one in u and hence compatible with the Euler capital allocation principle (cf. Paulusch, 2017). In contrast to $g(u)$, a Taylor polynomial of degree two (or higher) is not homogeneous of degree one, and a Taylor polynomial of degree one could not approximate the original function up to second-order partial derivatives.

For the special case that $(X_1, ..., X_n)^{\mathrm{T}}$ follows an elliptical distribution, section 3.3 shows that the sensitivity-implied tail-correlation matrix coincides with the Pearson correlation matrix. For non-elliptical distributions, the diagonal elements of the partial-derivative implied tail-correlation matrix can differ from one, as examples in section 4 will demonstrate.

3.3 Example: Elliptical distribution

Assume that the random vector $X = (X_1, ..., X_n)^{\mathrm{T}}$ follows an elliptical distribution and $x \in (0, \infty)^n$. Then, the true aggregate risk measurement as defined in (3) can be determined as[11]

$$
\begin{aligned}
f_{\varrho,X}(u) &= \varrho\left(\sum_{i=1}^{n} u_i \cdot X_i\right) - \mathbb{E}\left(\sum_{i=1}^{n} u_i \cdot X_i\right) = \sqrt{\sum_{i,j=1}^{n} \varrho_{ij}^{(P)} u_i x_i u_j x_j} \\
&= \sqrt{(u \circ x)^{\mathrm{T}} R_P \, (u \circ x)},
\end{aligned}
\tag{13}
$$

where the x_i are defined as in (4), and $R_P = \left(\varrho_{ij}^{(P)}\right)_{i,j=1}^{n}$ is the Pearson correlation matrix. Taking the square of the terms in Eq. (13) implies

$$
f_{\varrho,X}^2(u) = (u \circ x)^{\mathrm{T}} R_P \, (u \circ x) = \sum_{i=1}^{n} \sum_{j=1}^{n} \varrho_{ij}^{(P)} u_i x_i u_j x_j \tag{14}
$$

Differentiating the left-hand side and right-hand side of Eq. (14) with respect to u_k, $k \in \{1, ..., n\}$, implies

$$
\frac{\partial}{\partial u_k} f_{\varrho,X}^2(u) = 2x_k \sum_{j=1}^{n} \varrho_{kj}^{(P)} u_j x_j \tag{15}
$$

Differentiating both sides of Eq. (15) again with respect to u_ℓ, $\ell \in \{1, ..., n\}$ implies

$$
\begin{aligned}
&\frac{\partial^2}{\partial u_k \partial u_\ell} f_{\varrho,X}^2(u) = 2x_k x_\ell \varrho_{k\ell}^{(P)} \\
\Rightarrow \quad &\varrho_{k\ell}^{(P)} = \frac{1}{2x_k x_\ell} \frac{\partial^2}{\partial u_k \partial u_\ell} f_{\varrho,X}^2(u)
\end{aligned}
\tag{16}
$$

Hence, for elliptically distributed risks, all entries of the sensitivity-implied tail-correlation matrix coincide with those of the Pearson correlation matrix.

[11] Cf. McNeil et al. (2015, pp. 295), Devineau and Loisel (2009, section 4.2).

4 Numerical examples

4.1 Set-up

We consider an insurance company with $n = 3$ lines of business (lobs). The scalars u_1, u_2 and u_3 represent the volume of lob i in terms of the number of insurance contracts. We assume that the u_i are scaled, for example, in 100,000 contracts such that we may disregard the integer restriction. Moreover, we assume that the diversification within each lob does not vary in u_i such that the claims costs of lob i are modeled by $u_i \cdot X_i$. In line with Solvency II regulations, risk is measured by the 99.5% Value-at-Risk of unexpected losses.

We will consider classes of risk distributions for which the aggregate claims costs,

$$\sum_{i=1}^{n} u_i X_i, \tag{17}$$

have a distribution function with an analytical representation.[12] We thereby do not need to make use of Monte Carlo simulations, and all our reported results—with respect to $f_{\varrho,X}(u)$ or $g(u)$, including all relevant sensitivities—are thus unaffected by sampling error.

The connections between the volume u_i and the premium p_i of each lob $i \in \{1, ..., n\}$ are determined by an isoelastic demand function,[13]

$$u_i(p_i) = n_i \cdot p_i^{\epsilon_i}, \tag{18}$$

[12] Appendix D explains that the distribution function of the random variable in line (17) has an analytical representation if $X_1, ..., X_n$ are independent and gamma distributed random variables or if the random vector $(X_1, ..., X_n)^{\mathrm{T}}$ follows the so-called multivariate mixed-gamma distribution, as introduced by Furman et al. (2020).

[13] To simplify the notation, p_i is also scaled. If u_i are specified per 100,000 contracts, p_i is the premium income per 100,000 contracts.

where $n_i > 0$ calibrates demand to market size and $\epsilon_i < -1$ is the price elasticity of demand which is constant in p_i. We consider a representative insurer whose objective is to maximize its economic value added (EVA).[14] In our model, the insurer's EVA is the expected profit minus the cost of capital, which is modeled by a hurdle rate r_h times the 99.5% Value-at-Risk of the aggregate risk. In our baseline calibration, we set $\epsilon_i = -9$ for all lobs i,[15] and $r_h = 5\%$.[16]

On the one hand, we consider the EVA in connection with the risk measurement based on the true multivariate risk distribution:

$$\begin{aligned} \mathrm{EVA}_{\mathrm{true}}(u) &= \sum_{i=1}^{n} u_i \cdot (p_i(u_i) - \mathbb{E}[X_i]) - r_h \cdot f_{\varrho,X}(u) \\ &= \sum_{i=1}^{n} u_i \cdot (p_i(u_i) - \mathbb{E}[X_i]) - r_h \cdot \mathrm{VaR}_{99.5\%}\left(\sum_{i=1}^{n} u_i \cdot (X_i - \mathbb{E}[X_i])\right) \end{aligned} \tag{19}$$

with $p_i(u_i)$ denoting the inverse of the demand function in Eq. (18). We call the portfolio u which maximizes $\mathrm{EVA}_{\mathrm{true}}(u)$ the "true optimal portfolio". On the other hand, we identify which portfolio u an insurer chooses if the risk measurement is conducted in connection with a tail-correlation matrix R, i.e. the portfolio maximizing

$$\begin{aligned} \mathrm{EVA}_{R}(u) &= \sum_{i=1}^{n} u_i \cdot (p_i(u_i) - \mathbb{E}[X_i]) - r_h \cdot g(u) \\ &= \sum_{i=1}^{n} u_i \cdot (p_i(u_i) - \mathbb{E}[X_i]) - r_h \cdot \sqrt{(u \circ x)^{\mathrm{T}} R \, (u \circ x)} \end{aligned} \tag{20}$$

[14] The objective is analogous to the analysis of Chen et al. (2019). It can be justified by assuming that the insurer jointly decides on its level of equity capital and on the volumes $u_1, ..., u_n$; the regulatory capital requirement is based on VaR and binding. The objective can easily be modified to a situation in which the insurer sticks to a fixed capital requirement ratio (in terms of equity capital over capital requirement). In the context of Solvency II regulations, this ratio is relevant when insurers transmit information about their solvency level, cf. Gatzert and Heidinger (2020). In related analyses, the Economic Value Added has been employed by Stoughton and Zechner (2007) and Diers (2011).

[15] According to the empirical results of Yow and Sherris (2008, p. 318), this may reflect the price elasticity of compulsory third party or motor insurance.

[16] Zanjani (2002, p. 297) estimates that the discounted cost of holding capital is 5% in commercial automobile insurance.

The chosen set-up allows us to distinguish the distortions caused by the function $g(u)$ in terms of EVA and in terms of the insurer's safety level. The insurer's safety level is measured by the true VaR confidence level which corresponds to $g(u)$, i.e. the solution $\tilde{\alpha}$ of

$$\mathrm{VaR}_{1-\tilde{\alpha}}\left(\sum_{i=1}^{n} u_i \cdot (X_i - \mathbb{E}[X_i])\right) = g(u) \tag{21}$$

4.2 Relevance of first-order sensitivities

This section demonstrates that an inappropriate calibration of the matrix R can lead to biased first-order sensitivities of the aggregate risk measurement and can induce a suboptimal portfolio.

The basic losses of the three lobs are modeled by the stochastically independent and gamma distributed random variables $\tilde{X}_1$, $\tilde{X}_2$ and $\tilde{X}_3$. Specifically, we assume that $\tilde{X}_1 \sim \Gamma(\frac{1}{3}, \frac{2}{3})$, $\tilde{X}_2 \sim \Gamma(2, 2)$ and $\tilde{X}_3 \sim \Gamma(1, 2)$, where $\Gamma(\gamma, \vartheta)$ denotes the gamma distribution with shape parameter γ and rate parameter ϑ. In addition, lobs 1 and 3 are exposed to a common risk factor $Y \sim \Gamma(1, 1)$, which is independent from the $\tilde{X}_i$. The total claims costs of the three lobs are $X_1 = \tilde{X}_1 + 0.5Y$, $X_2 = \tilde{X}_2$ and $X_3 = \tilde{X}_3 + 0.5Y$.[17]

[17] The expected claims costs of all three lobs are 1, e.g. we calculate $\mathbb{E}[X_1] = \mathbb{E}[\tilde{X}_1] + 0.5\mathbb{E}[Y] = \left(\frac{1}{3}\right) / \left(\frac{2}{3}\right) + 0.5 \cdot 1/1 = 1$. Moreover, the variances are $\mathrm{var}[X_1] = \mathrm{var}[\tilde{X}_1] + 0.5^2\mathrm{var}[Y] = \left(\frac{1}{3}\right) / \left(\frac{2}{3}\right)^2 + 0.5^2 \cdot 1/1^2 = 1$ and $\mathrm{var}[X_2] = \mathrm{var}[X_3] = 0.5$. The risks of lob 1 have a relatively heavy tail, the risks of lob 2 have a relatively light tail. The ratios of the 99.5% VaR and the 90% VaR are 4.53 for lob 1, 2.87 for lob 2 and 3.06 for lob 3. For comparison, Bernard et al. (2018, p. 847) assume the distribution $200 \cdot \mathrm{LogNormal}(0, 1)$ for aggregate non-life insurance risks. This implies a corresponding VaR ratio of 5.88. For the aggregate market risk, Bernard et al. (2018, p. 847) assume a normal distribution, for which the corresponding VaR ratio is 2.01. This value is achieved by the gamma distribution when setting the shape parameter to infinity.

The vector of stand-alone capital requirements is

$$x = \begin{pmatrix} 4.56 \\ 2.72 \\ 2.25 \end{pmatrix} \tag{22}$$

and the Pearson correlation matrix is

$$R_P = \begin{pmatrix} 1 & 0 & 0.35 \\ 0 & 1 & 0 \\ 0.35 & 0 & 1 \end{pmatrix} \tag{23}$$

The insurer's current portfolio is $u = \mathbb{1}_3 = (1, 1, 1)^{\mathrm{T}}$ and the corresponding aggregate risk measurement—based on the true multivariate risk distribution—is

$$f_{\varrho,X}(\mathbb{1}_3) = \mathrm{VaR}_{0.995}(X_1 + X_2 + X_3) - \mathbb{E}[X_1 + X_2 + X_3] = 5.93 \tag{24}$$

The true sensitivities of the aggregate risk measurement—i.e., the Euler allocation—are obtained as

$$\nabla f_{\varrho,X}(u)\Big|_{u=\mathbb{1}_3} = \begin{pmatrix} 3.79 \\ 0.64 \\ 1.50 \end{pmatrix} \tag{25}$$

According to line (25), the first lob has the strongest impact on the insurer's aggregate risk. The third lob follows next—due to its positive correlation with the risks of the first lob. The second lob is less influential due to its independence from the other risks, even though its stand-alone risk is higher than that of lob 3.

Table 1: Sensitivity-implied and VaR-implied tail-correlation matrices.

Sensitivity-implied			
	Lob 1	Lob 2	Lob 3
Lob 1	1.118	-0.165	0.122
Lob 2	-0.165	0.849	-0.066
Lob 3	0.122	-0.066	1.600

VaR-implied (Mittnik, 2014)	Pairwise			Exact			Overidentified		
	Lob 1	Lob 2	Lob 3	Lob 1	Lob 2	Lob 3	Lob 1	Lob 2	Lob 3
Lob 1	1.000	-0.184	-0.123	1.000	0.268	-0.123	1.000	-0.128	-0.055
Lob 2	-0.184	1.000	-0.177	0.268	1.000	-0.177	-0.128	1.000	-0.063
Lob 3	-0.123	-0.177	1.000	-0.123	-0.177	1.000	-0.055	-0.063	1.000

VaR-implied (Devineau and Loisel, 2009)	Minimal standard		
	Lob 1	Lob 2	Lob 3
Lob 1	1.000	0.041	0.034
Lob 2	0.041	1.000	0.020
Lob 3	0.034	0.020	1.000

To calibrate the VaR-implied tail-correlation matrix in line with Mittnik (2014), we need to choose a set of $\ell \geq 3$ calibration portfolios. We consider three of those sets. Firstly, we conduct a *pairwise* calibration based on all equally-weighted two-risk portfolios. Hence, we set $w_1 = (1, 1, 0)^{\mathrm{T}}$, $w_2 = (1, 0, 1)^{\mathrm{T}}$ and $w_3 = (0, 1, 1)^{\mathrm{T}}$. Secondly, we consider an *exact* calibration based on $w_1 = (1, 1, 1)^{\mathrm{T}}$, $w_2 = (1, 0, 1)^{\mathrm{T}}$ and $w_3 = (0, 1, 1)^{\mathrm{T}}$. Thirdly, we conduct an *overidentified* calibration based on all equally-weighted two and three-asset portfolios, i.e. with $w_1 = (1, 1, 0)^{\mathrm{T}}$, $w_2 = (1, 0, 1)^{\mathrm{T}}$, $w_3 = (0, 1, 1)^{\mathrm{T}}$ and $w_4 = (1, 1, 1)^{\mathrm{T}}$. In addition, we calculate the "minimal standard" VaR-implied tail-correlation matrix as proposed by Devineau and Loisel (2009) as well as the sensitivity-implied tail-correlation matrix, both with the calibration portfolio being the insurer's current portfolio $u = (1, 1, 1)^{\mathrm{T}}$. Table 1 presents all five calculated tail-correlation matrices.

Table 2 reports the aggregate risk of the current portfolio, $u = \mathbb{1}_3$, and the Euler allocations in connection with the VaR-implied and sensitivity-implied tail-correlation matrices. The results for VaR-implied matrices in the sense of Mittnik (2014) clearly depend on

Table 2: Chosen portfolios of the model insurer. The risk measurement is conducted based on the true multivariate risk measurement, the sensivitivy-implied tail-correlation matrix or a VaR-implied tail-correlation matrix.

Type of calculation	True distribution	Sensitivity-implied	VaR-implied Pairwise	Exact	Overident.	Min. Std.
Aggregate risk measurement	5.93	5.93 (±0%)	4.90 (−17%)	5.93 (±0%)	5.31 (−11%)	5.93 (±0%)
Euler allocation						
Lob 1	3.79	3.79 (±0%)	3.53 (−7%)	3.86 (+2%)	3.52 (−7%)	3.65 (−4%)
Lob 2	0.64	0.64 (±0%)	0.82 (+28%)	1.62 (+152%)	1.02 (+59%)	1.35 (+110%)
Lob 3	1.50	1.50 (±0%)	0.55 (−63%)	0.46 (−70%)	0.77 (−49%)	0.93 (−38%)
Chosen portfolio						
Lob 1	1.00	1.00	1.13	1.02	1.10	1.03
Lob 2	1.00	1.00	1.01	0.76	0.93	0.82
Lob 3	1.00	1.00	1.36	1.35	1.27	1.20
True EVA	0.412	0.412 (±0%)	0.402 (−2.3%)	0.397 (−3.6%)	0.406 (−1.6%)	0.406 (−1.5%)
True VaR confidence level	0.50 %	0.50%	1.30%	0.77%	0.92 %	0.60 %

the set of calibration portfolios. Moreover, based on a VaR-implied matrix, the aggregate risk of the current portfolio can be underestimated by 5% (overidentified calibration) or even 12% (pairwise calibration).[18] In connection with all considered calibrations, the VaR-implied tail-correlation matrices lead to substantially biased Euler allocations, which erroneously rank the second lob to be more influential for the insurer's aggregate risk than the third lob. In contrast, according to Proposition 1, the use of the sensitivity-implied tail-correlation matrix leads to an accurate measurement of the aggregate risk and Euler allocations.

In terms of the EVA analysis, we set the demand function parameters to $n_1 = 13.751$, $n_2 = 3.837$ and $n_3 = 5.539$. These parameter values imply that the true EVA-optimal strategy is $u = \mathbb{1}_3$. The same strategy maximizes the EVA in line (20) if R is the

[18] As explained in section 3.1, the VaR-implied tail-correlation matrix leads to a correct risk assessment of the current portfolio only if the number of calibration portfolios coincides with the number of correlation parameters and the current portfolio is one of the calibration portfolios.

sensitivity-implied tail-correlation matrix. However, the distorted risk measurement based on the VaR-implied matrices lead the insurer away from the truly optimal strategy. For example, in the case of a pairwise calibration, the insurer chooses $u_{\text{new}} = (1.13, 1.01, 1.36)^{\mathrm{T}}$. Based on the true multivariate risk distribution, the aggregate risk of the chosen portfolio is $f_{\varrho,X}(u_{\text{new}}) = 7.00$. The VaR-implied correlation matrix in connection with a pairwise calibration, however, understates the risk of this portfolio by $g(u_{\text{new}})/f_{\varrho,X}(u_{\text{new}}) - 1 = 5.59/7.00 - 1 = -20\%$. The true VaR-confidence level of the chosen strategy is clearly too high and amounts to 1.3%. In addition, the true EVA of the chosen portfolio, $\text{EVA}_{\text{true}}(u_{\text{new}})$, is 2.3% lower than the maximal EVA, i.e. $\text{EVA}_{\text{true}}(\mathbb{1}_3)$.

In the analyses so far, the calibrations of the tail-correlation matrices were centered at the true optimal portfolio, $u = \mathbb{1}_3$. Next, we study how insurance companies with different properties—and different true optimal portfolios—choose their portfolios if their risk measurement is based on the tail-correlation matrices calibrated at $\mathbb{1}_3$. We modify seven parameters—namely the values of the demand function parameters, n_1, n_2, n_3, ϵ_1, ϵ_2 and ϵ_3 as well as the hurdle rate r_h—by multiplying them with scalars. To this end, we randomly choose seven scalars as independent realizations of uniform random variables on the interval 0.6 to 1.4. This process is executed 500 times to generate 500 heterogeneous insurance companies. As shown by Table 3, the true optimal portfolios of the 500 insurers differ from $\mathbb{1}_3$ with a root mean squared error (RMSE) of 0.302 on average across the 500 insurers. The risk of the true optimal portfolios is not measured accurately by any of the tail-correlation matrices. Nevertheless, the sensitivity-implied matrix guides insurers to a portfolio which achieves almost the same EVA as the true optimal portfolio (cf. Table 3: for 90% of insurers the relative loss in EVA does not exceed 0.01%). In contrast, measuring risk based on VaR-implied matrices goes along with a

Table 3: Chosen portfolios of 500 randomly parameterized insurers. The risk measurement is conducted based on the true multivariate risk measurement, the sensitivity-implied tail-correlation matrix or a VaR-implied tail-correlation matrix. The table reports the means across the 500 insurers as well as the 10% and 90% percentiles.

Type of calculation	True distribution	Sensitivity-implied	VaR-implied Pairwise	Exact	Overident.	Min. Std.
RMSE between chosen portfolio and $\mathbb{1}_3$						
Mean	0.302	0.300	0.372	0.397	0.354	0.353
$p_{10\%}$	0.124	0.124	0.149	0.211	0.145	0.170
$p_{90\%}$	0.514	0.512	0.668	0.639	0.630	0.574
Relative loss in EVA of chosen portfolio versus "true" optimal portfolio						
Mean	0.00%	0.00%	2.08%	3.12%	1.36%	1.32%
$p_{10\%}$	0.00%	0.00%	0.77%	1.37%	0.50%	0.51%
$p_{90\%}$	0.00%	0.01%	3.78%	5.33%	2.54%	2.35%
True VaR confidence level						
Mean	0.50%	0.50%	1.23%	0.76%	0.90%	0.60%
$p_{10\%}$	0.50%	0.50%	1.00%	0.59%	0.77%	0.52%
$p_{90\%}$	0.50%	0.50%	1.48%	0.95%	1.04%	0.70%

considerable loss in EVA. For example, the "exact" calibration of the VaR-implied matrix leads to a relative loss in EVA of 3.12% on average and of 5.33% or higher for those 10% of insurers with the highest loss. Moreover, when using the sensitivity-implied matrix, the true VaR confidence level of insurers' aggregate risk is close to the supposed value of 0.5%. With respect to VaR-implied matrices, each of the four calibration methods guides more than 90% of insurers to strategies with a true VaR confidence level above 0.5%. Figure 1 visualizes the loss in EVA and the true VaR confidence level of the portfolios that the 500 insurers choose based on the tail-correlation matrices. The figure depicts that the sensitivity-implied matrix leads to substantially smaller distortions than any of the VaR-implied matrices.

4.3 Relevance of second-order sensitivities

This section studies the implications of biased second-order sensitivities of the aggregate risk measurement based on VaR-implied tail-correlation matrices. The three lobs' claims

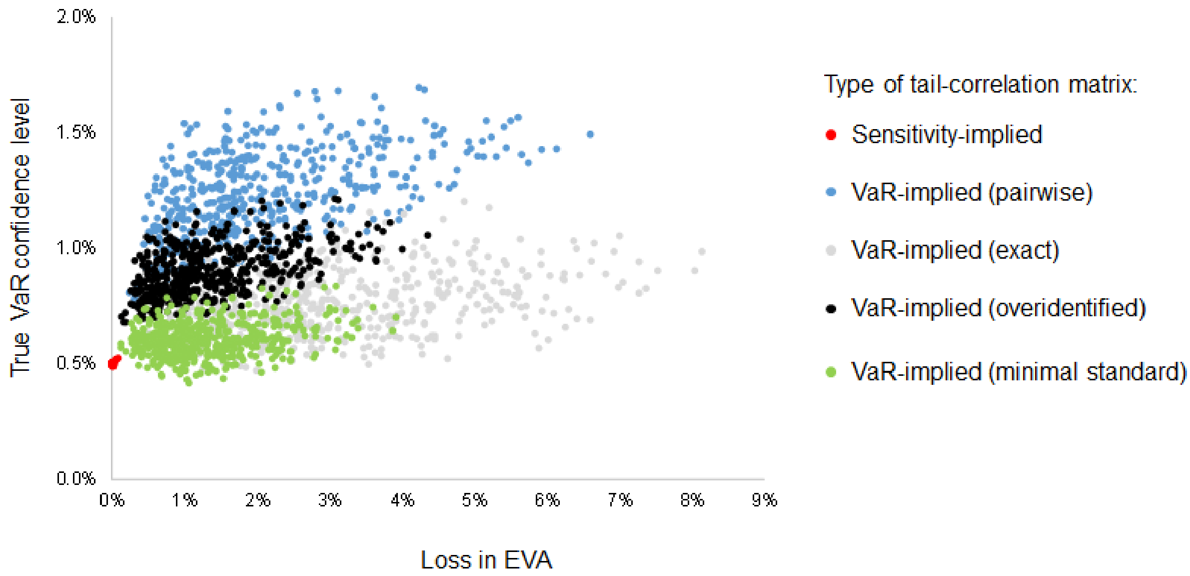

Figure 1: Loss in EVA and true VaR confidence level for 500 randomly parameterized insurers. The risk measurement is conducted based on the sensitivity-implied tail-correlation matrix or a VaR-implied tail-correlation matrix. The results show that portfolio optimization in connection with the sensitivity-implied tail-correlation matrix hardly induces distortions in terms of the VaR confidence level or the achieved EVA.

costs, X_1, X_2 and X_3, now follow a mixed-gamma distribution with the parameters defined in Table 4.[19] With a large weight in terms of p_κ, the distribution consists of $n = 3$ independent and identically distributed risks. However, conditioning on a high aggregate loss, the risks X_1 and X_2 are negatively correlated. In this set-up, the marginal capital requirement of X_1 decreases when increasing the exposure to X_1. Hence, the Hesse matrix of $f_{\varrho,X}$ with respect to u has negative entries on the diagonal:

$$H_u f_{\varrho,X}(u) = \begin{pmatrix} -3.850 & 3.632 & 0.218 \\ 3.632 & -3.850 & 0.218 \\ 0.218 & 0.218 & -0.437 \end{pmatrix}$$

[19]Appendix D provides more details about this distribution.

Table 4: Parameters of the mixed-gamma distribution.

i	ϑ_i	γ_{k_1}	γ_{k_2}	γ_{k_3}
1	0.5	0.5	9.5	0.5
2	0.5	0.5	0.5	9.5
3	0.5	0.5	4.5	4.5
p_κ		0.99	0.005	0.005

The aggregate Value-at-Risk can be reduced by shifting the exposures from $u = (1, 1, 1)^{\mathrm{T}}$ to $u = (1 + h, 1 - h, 1)^{\mathrm{T}}$ for a small value of h.[20] We embed this distribution into the EVA-optimization problem as studied in section 4.2. By setting $n_1 = n_2 = 128.082$ and $n_3 = 90.209$, all first-order derivatives of the function $\mathrm{EVA}_{\mathrm{true}}(u)$ are zero at $u = \mathbb{1}_3$. The Hesse matrix of $\mathrm{EVA}_{\mathrm{true}}(u)$ is indefinite at $u = \mathbb{1}_3$ reflecting the fact that it is a saddle point, as illustrated on the left side of Figure 2. To keep the example graphically fully tractable, we assume from now on that $u_3 = 1$ is fixed and only u_1 and u_2 are decision variables. Then the function $\mathrm{EVA}_{\mathrm{true}}(u_1, u_2|u_3 = 1)$ has a global maximum at $(u_1, u_2) = (1.8365, 0.5998)$ and, due to symmetry, another global maximum at $(0.5998, 1.8365)$; cf. points B and B' in Figure 2.

We calibrate two tail-correlation matrices, R_1 and R_2, both with the calibration portfolio $u = \mathbb{1}_3$. R_1 is the sensitivity-implied tail-correlation matrix. R_2 is calibrated such that the function $g(u)$ reflects the true first-order sensitivities:

$$\nabla_u g\left(\mathbb{1}_3\right) = \nabla_u f\left(\mathbb{1}_3\right) \tag{26}$$

[20] This can be seen by approximating $f_{\varrho,X}(u)$ by a Taylor polynomial of degree 2 and noting that $\partial/\partial u_1 f(u) = \partial/\partial u_2 f(u)$ at $u = (1, 1, 1)^{\mathrm{T}}$.

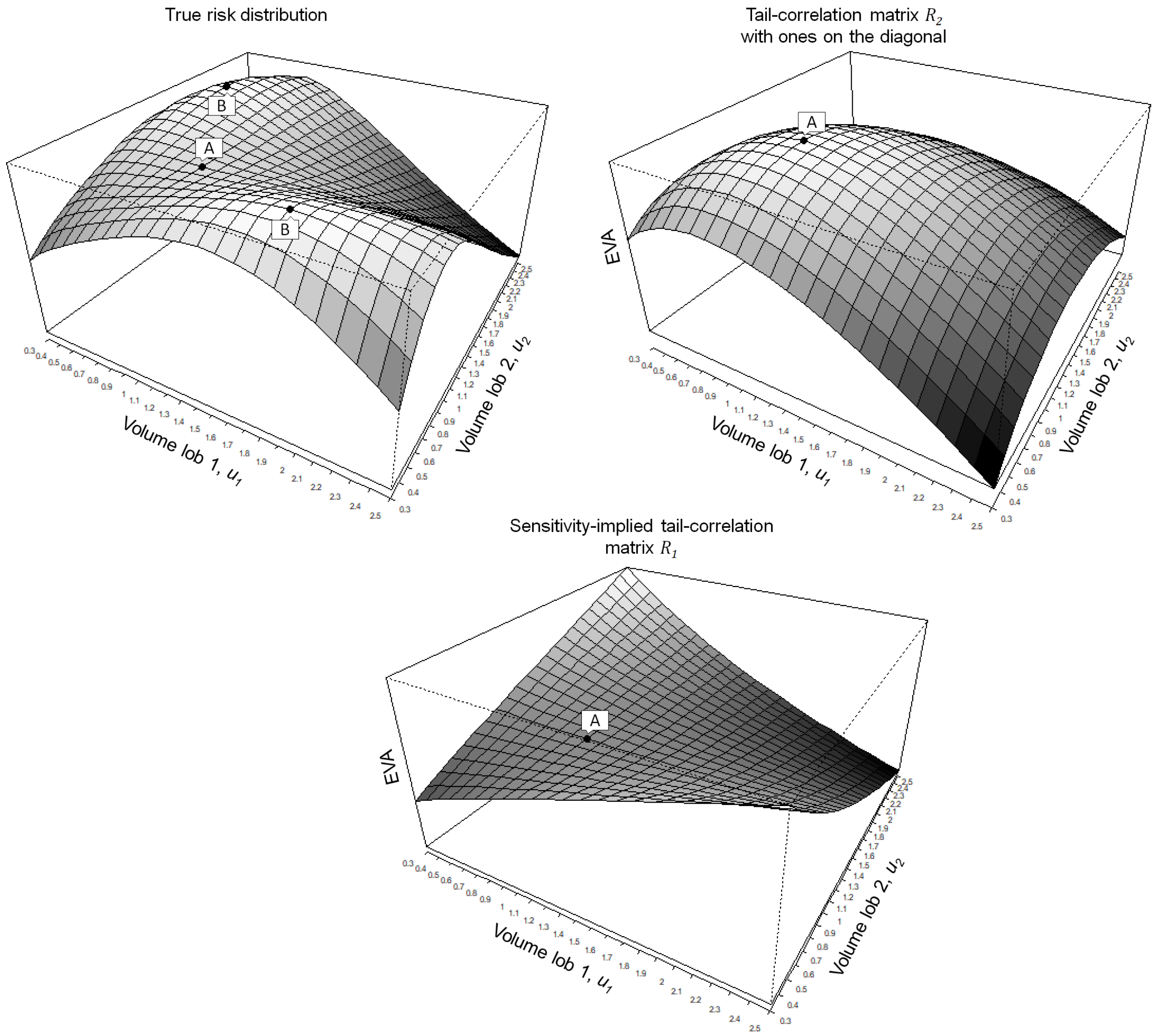

Figure 2: EVA based on volumes u_1 and u_2 and for fixed $u_3 = 1$. Point A reflects $u_1 = u_2 = 1$; points B and B' are optimal based on true risk distribution.

In accordance with the traditional view of tail-correlation matrices, we restrict R_2 to be a symmetric matrix with ones on its diagonal. The three correlation parameters in R_2 are thereby uniquely defined by (26).[21]

The right side of Figure 2 depicts the EVA function in connection with the matrix R_2. By construction, all first-order derivatives of this EVA function are zero at $u = \mathbb{1}_3$. However,

[21]Specifically, the three correlation parameters are the solution of a linear equation system. As shown by Eq. (49) in Appendix E, $\nabla_u g\left(\mathbb{1}_3\right)$ is linear in the correlation parameters when fixing $\sqrt{x^{\mathrm{T}} R_2\, x}$ at $f_{\varrho,X}\left(\mathbb{1}_3\right)$.

the Hesse matrix of this EVA function is negative definite and hence, this function has a global maximum at $u = (1, 1, 1)^{\mathrm{T}}$—in contrast to the true EVA function.

The lower part of Figure 2 shows the EVA function in connection with the sensitivity-implied tail-correlation matrix R_1. This EVA function correctly approximates the true EVA function at the calibration portfolio $u = \mathbb{1}_3$ and hence, the company does not mistake the saddle point for an optimum.

5 Comparison of VaR-implied and sensitivity-implied tail correlation matrices

5.1 Conceptual comparison

A VaR-implied tail-correlation matrix can be viewed as a tool to approximate the true risk measurement $f_{\varrho,X}(u)$ as defined in line (3). Suppose that the VaR-implied tail-correlation matrix R minimizes (7) for the calibration portfolios w_1,...,w_ℓ. In connection with this matrix R, the function $g(u)$, as defined in (5), approximates $f_{\varrho,X}(u)$ in the following sense:

- $g(w_i)$ is close to $f_{\varrho,X}(w_i)$ for all calibration portfolios w_i, $i \in \{1, ..., \ell\}$. If the calibration is "exact", i.e. if $\ell = n(n-1)/2$, then $g(w_i) = f(w_i)$ for all $i \in \{1, ..., \ell\}$.
- g exactly reflects the risk of all one-asset portfolios. Technically, if all but one entries of u are zero, then $g(u) = f(u)$.[22]

[22]This property follows directly from the assumption that all diagonal elements of the VaR-implied tail-correlation matrix R are equal to one.

- Except for $n = 2$ risks, the VaR-implied tail-correlation matrix is in general not unique, but it can vary depending on the chosen set of calibration portfolios, $w_1, ..., w_\ell$. The choice of calibration portfolios affects the matrix R even if the calibration is conducted based on complete information about the multivariate risk distribution (cf. section 4.2).

- Mittnik (2014, p. 71 f.) presents strategies to modify R in a such way that it is positive semidefinite or strictly positive definite, which is considered as desirable when using R in the context of mean-variance portfolio optimization.

In connection with the sensitivity-implied tail-correlation matrix R, the function $g(u)$ in (5) approximates $f_{\varrho,X}(u)$ with the following properties:

- Once the single calibration portfolio has to be chosen, the sensitivity-implied tail-correlation matrix is unique.

- All first and second-order sensitivities of $f_{\varrho,X}(u)$ are correctly reflected by $g(u)$ at the calibration portfolio.

In sum, both types of tail-correlation matrices provide portfolio managers with a deterministic risk aggregation approach, cf. formula (5), which approximates the true risk measurement. The sensitivity-implied matrix approximates the risk measurement locally at the calibration portfolio. As indicated by the examples in section 4.2, the risk measurement is useful for portfolios in an environment around the calibration portfolio. The risk of one-asset portfolios can be misstated by the use of the sensitivity-implied matrix, given that this matrix does not necessarily have ones on the diagonal. Nevertheless, one should keep in mind that forcing R to have ones on the diagonal can induce a misleading view of a balanced portfolio's aggregate risk, as the example in section 4.3 has demonstrated.

The sensitivity-implied tail-correlation matrix is not necessarily positive semidefinite, which can have two consequences. Firstly, the approximation $g(u)$ is not necessarily concave in u. The missing concavity should not be viewed as a drawback in terms of portfolio optimization, since it allows $g(u)$ to reflect the shape of $f_{\varrho,X}(u)$. As illustrated in section 4.3, $g(u)$ thereby allows the portfolio manager to distinguish a saddle point from a local optimum. Secondly, there can be portfolios u for which $g(u)$ is not defined in real numbers, since the radicand in (5) can be negative. Again, this artifact is not in contrast with the purpose of the sensitivity-implied matrix of approximating $f_{\varrho,X}(u)$ in a neighborhood of the calibration portfolio, and hence it should not be suppressed by altering R. $g(u)$ being not defined in real numbers may be considered as a signal that the portfolio u is outside this neighborhood.

5.2 Calibration

In many applications, the calibration of both types of tail-correlation matrices—VaR-implied and sensitivity-implied—might start from a fitted multivariate distribution. Under this prerequisite, the calibration of the sensitivity-implied tail-correlation matrix is conceptually not more challenging than the calibration of a VaR-implied tail-correlation matrix. The estimation of a VaR-implied tail-correlation matrix from a finite sample requires estimates of the true risk measurement of all calibration portfolios, i.e. $\widehat{f_{\varrho,X}(w_1)}$, ..., $\widehat{f_{\varrho,X}(w_\ell)}$. Regarding the risk measure $\mathrm{VaR}_{1-\alpha}$, a direct estimation of these figures can only be possible if the confidence level $1-\alpha$ is not too close to one and the holding period is short. In terms of Solvency II, for example, the confidence level is 99.5% and the holding period is one year. Hence, a direct estimation of the $\widehat{f_{\varrho,X}(w_i)}$ would require data series with a length of several hundred years. Instead of such a direct estimation, it might often be more realistic to fit a multivariate risk distribution to the data and estimate the VaR-implied tail-correlation matrix from that distribution.

Regarding the sensitivity-implied tail-correlation matrix, we see two possible estimation strategies which both start from a fitted multivariate risk distribution. Firstly, Furman et al. (2020) show that any continuous distribution with positive support can be approximated arbitrarily well by a multivariate mixed-gamma distribution.[23] Based on the mixed-gamma distribution, the sensitivity-implied tail-correlation matrix can be calculated by numerical differentiation, as explained in Appendix D. Secondly, the estimation of the sensitivity-implied tail-correlation matrix can be based on Monte-Carlo simulations. This strategy seems to be particularly promising if the risk measure Conditional Value-at-Risk (CVaR) is applied, given that first-order derivatives of $f_{\varrho,X}(u)$ can then be efficiently estimated from simulations (cf. Hong and Liu, 2009). Second-order derivatives can be obtained numerically, i.e. by simulating the random vectors $(X_1, ..., X_n)^{\mathrm{T}}$, $((1+h)X_1, ..., X_n)^{\mathrm{T}}$, $(X_1, (1+h)X_2, ..., X_n)^{\mathrm{T}}$ etc. for some small $h > 0$.

6 Conclusion

This paper demonstrates that a risk measurement in connection with traditional VaR-implied tail-correlation matrices can state the impact of portfolio changes substantially differently than an assessment based on the true multivariate risk distribution. In particular, the traditional notion of tail-correlation matrices having ones on their diagonal

[23] Furman et al. (2020) show in Theorem 4 that the class of mixed-gamma distributions is dense in the class of continuous multivariate distributions with non-negative supports. More precisely, for any random vector in the latter class, a sequence of mixed-gamma distributed random vectors can be constructed which converges in distribution to the given random vector.

can make it impossible to fit them in accordance with the true distribution. Those misstatements can distort portfolio management decisions in terms of risk and return.

We propose so-called "sensitivity-implied" tail-correlation matrices, which approximate the risk measurement based on the true multivariate risk distribution up to second-order derivatives with respect to exposures. We see several areas of application.

In the context of regulation, the proposed method may help to circumvent moral hazard effects arising from misstated marginal capital requirements, as empirically detected by Chen et al. (2019). Our example in section 4.2 indicates that the matrix R would not have to be calibrated for every insurer. Instead, the sensitivity-implied tail-correlation leads to a relatively stable risk measurement and steering signals even if insurers attain different portfolios than the one which is assumed in the calibration of R. The use of a VaR-implied tail-correlation matrix goes along with larger and more varying distortions if the matrix is used by a heterogeneous group of insurers.

In the context of a firm's internal economic capital assessment, the use of the correlation-based risk aggregation is sometimes called the "hybrid approach" (Hull, 2018, p. 594). In comparison with a risk aggregation based on a Monte-Carlo simulation, the correlation-based approach facilitates the risk measurement process, since changes in the univariate risk assessments do not require new simulations of the entire firm. Once the matrix R has been calibrated, various methods (including scenario analyses, expert surveys, etc.) can be used for the measurement of the univariate risks. Given that the risk assessment based on the sensitivity-implied tail-correlation matrix is relatively stable when exposures deviate from the calibration portfolio (as indicated by section 4.2), the matrix R does not need to be immediately recalibrated when the firm's portfolio or the assessments of univariate risks moderately change.

Finally, the proposed tail-correlation matrix can be helpful for portfolio optimization in general when risk is to be measured by a homogeneous risk measure, such as VaR oder CVaR. Approximating the true risk measurement $f_{\varrho,X}(u)$ using the correlation-based risk measurement $g(u)$ may simplify the optimization problem. For instance, optimization problems with a VaR or CVaR constraint thereby become structurally identical to mean-variance portfolio optimization.

Appendix

A Homogeneous approximation function

The mathematical core of the sensitivity-implied tail-correlation matrix is the observation that any function f which is homogeneous of degree one and twice continuously differentiable can be locally approximated by a function which corresponds to the definition in (5) and is homogeneous of degree one. In this section we state our result for a generic function f.

Theorem 1. *Let $n \in \mathbb{N}$, $U \subseteq \mathbb{R}^n$ be open, $x \in U$, and the function $f : U \to \mathbb{R}$ be homogeneous of degree one and twice continuously differentiable. Let $f(x) > 0$. Then, the matrix $R = R(x) = (\varrho_{k\ell})_{k,\ell=1}^n$ defined by*

$$\varrho_{k\ell} = \varrho_{k\ell}(x) = \frac{1}{2} \frac{\partial^2}{\partial x_k \partial x_\ell} f^2(x) \tag{27}$$

is symmetric. Writing

$$f_x(u) = f(u \circ x) \tag{28}$$

and

$$g_x(u) = \sqrt{(u \circ x)^{\mathrm{T}}\, R(x)\, (u \circ x)}, \tag{29}$$

the following holds:

$$\begin{aligned} g_x(\mathbb{1}_n) &= f_x(\mathbb{1}_n), & (30)\\ \frac{\partial}{\partial u_\ell} g_x(\mathbb{1}_n) &= \frac{\partial}{\partial u_\ell} f_x(\mathbb{1}_n), \quad 1 \le \ell \le n, & (31)\\ \frac{\partial^2}{\partial u_k \partial u_\ell} g_x(\mathbb{1}_n) &= \frac{\partial^2}{\partial u_k \partial u_\ell} f_x(\mathbb{1}_n), \quad 1 \le k, \ell \le n. & (32) \end{aligned}$$

B Proof of Theorem 1

We write $\partial_\ell = \partial/\partial x_\ell$ and $\partial_\ell = \partial/\partial u_\ell$, applying to functions of the variable x, or u, respectively. Note that the chain rule implies

$$\partial_\ell f^2(x) = 2f(x)\partial_\ell f(x) \tag{33}$$

and

$$\varrho_{k\ell}(x) = \frac{1}{2}\frac{\partial^2}{\partial x_k \partial x_\ell} f^2(x) = \frac{\partial}{\partial x_\ell}\left\{ f(x)\frac{\partial f}{\partial x_k}(x)\right\} = \partial_\ell \left\{ f(x)\partial_k f(x)\right\} \tag{34}$$

Schwarz's Theorem on the symmetry of second-order derivatives shows the symmetry of R, and the product rule implies

$$\begin{aligned}\partial_\ell \left\{ f(x)\sum_{k=1}^n x_k \partial_k f(x)\right\} &= \partial_\ell \left\{ \sum_{k=1}^n x_k \big(f(x)\partial_k f(x)\big)\right\} \\ &= \sum_{k=1}^n x_k \partial_\ell \left\{ f(x)\partial_k f(x)\right\} + f(x)\partial_\ell f(x)\end{aligned} \tag{35}$$

We use this and Euler's Theorem on homogeneous functions (cf. Tasche, 2008, p. 4), namely

$$\sum_{\ell=1}^n x_\ell \partial_\ell f(x) = f(x), \tag{36}$$

and derive

$$\begin{aligned}\big(g_x(\mathbb{1}_n)\big)^2 &= \sum_{k=1}^n \sum_{\ell=1}^n x_k x_\ell \varrho_{k\ell} \overset{(34)}{=} \sum_{k=1}^n \sum_{\ell=1}^n x_k x_\ell \partial_\ell \left\{ f(x)\partial_k f(x)\right\} \\ &= \sum_{\ell=1}^n x_\ell \left[\sum_{k=1}^n x_k \partial_\ell \left\{ f(x)\partial_k f(x)\right\}\right] \\ &\overset{(35)}{=} \sum_{\ell=1}^n x_\ell \left[\partial_\ell \left\{ f(x)\sum_{k=1}^n x_k \partial_k f(x)\right\} - f(x)\partial_\ell f(x)\right]\end{aligned}$$

$$\overset{(36)}{=} \sum_{\ell=1}^{n} x_\ell \left[\partial_\ell f^2(x) - f(x)\partial_\ell f(x)\right] \overset{(33)}{=} \sum_{\ell=1}^{n} f(x) x_\ell \partial_\ell f(x)$$

$$\overset{(36)}{=} f^2(x) = \big(f_x(\mathbb{1}_n)\big)^2$$

The assumption $f(x) > 0$ now implies (30). To prove equation (31), we note that the chain rule and (28) imply

$$\partial_\ell f_x(\mathbb{1}_n) = x_\ell \, \partial_\ell f(x) \tag{37}$$

We derive

$$\begin{aligned}
\partial_\ell g_x(\mathbb{1}_n) &\overset{(48)}{=} \frac{x_\ell}{g_x(\mathbb{1}_n)} \sum_{k=1}^{n} \varrho_{k\ell} x_k \overset{(30),\,(34)}{=} \frac{x_\ell}{f_x(\mathbb{1}_n)} \sum_{k=1}^{n} x_k \partial_\ell \left\{f(x)\partial_k f(x)\right\} \\
&\overset{(35)}{=} \frac{x_\ell}{f_x(\mathbb{1}_n)} \left[\partial_\ell \left\{ f(x) \sum_{k=1}^{n} x_k \partial_k f(x) \right\} - f(x)\partial_\ell f(x)\right] \\
&\overset{(36)}{=} \frac{x_\ell}{f_x(\mathbb{1}_n)} \left[\partial_\ell f^2(x) - f(x)\partial_\ell f(x)\right] \overset{(33)}{=} \frac{x_\ell \, f(x) \, \partial_\ell f(x)}{f_x(\mathbb{1}_n)} \overset{(37)}{=} \partial_\ell f_x(\mathbb{1}_n)
\end{aligned}$$

This is equation (31). To prove Equation (32), we write $\partial_{k\ell} = \partial^2/\big(\partial x_k \partial x_\ell\big)$ for functions of x and $\partial_{k\ell} = \partial^2/\big(\partial u_k \partial u_\ell\big)$ for functions of u. The chain rule and (28) imply

$$\partial_{k\ell} f_x^2(\mathbb{1}_n) = x_k x_\ell \partial_{k\ell} f^2(x) \tag{38}$$

We derive

$$\partial_{k\ell} g_x^2(\mathbb{1}_n) = 2 x_k x_\ell \varrho_{k\ell} \overset{(34)}{=} 2 x_k x_\ell \, \partial_\ell \left\{f(x)\partial_k \, f(x)\right\} \overset{(33)}{=} x_k x_\ell \partial_{k\ell} f^2(x) \overset{(38)}{=} \partial_{k\ell} f_x^2(\mathbb{1}_n) \tag{39}$$

We thereby have

$$\frac{1}{2} \, \partial_{k\ell} g_x^2(\mathbb{1}_n) = \frac{1}{2} \, \partial_{k\ell} f_x^2(\mathbb{1}_n)$$

$$\Longrightarrow \partial_\ell\big(g_x(\mathbb{1}_n)\cdot\partial_k g_x(\mathbb{1}_n)\big) = \partial_\ell\big(f_x(\mathbb{1}_n)\cdot\partial_k f_x(\mathbb{1}_n)\big)$$

$$\Longrightarrow \partial_\ell g_x(\mathbb{1}_n)\partial_k g_x(\mathbb{1}_n) + g_x(\mathbb{1}_n)\partial_{k\ell} g_x(\mathbb{1}_n) = \partial_\ell f_x(\mathbb{1}_n)\partial_k f_x(\mathbb{1}_n) + f_x(\mathbb{1}_n)\partial_{k\ell} f_x(\mathbb{1}_n)$$

$$\overset{(30),\,(31)}{\Longrightarrow} \partial_{k\ell} g_x(\mathbb{1}_n) = \partial_{k\ell} f_x(\mathbb{1}_n)$$

This finishes the proof.

C Proof of Proposition 1

Let the entries of the vector $\bar{x} \in \mathbb{R}^n$ be defined as $\bar{x}_i = x_i^{-1}$, and the open set $U \subseteq \mathbb{R}^n$ by

$$U = \{u \circ x \,|\, u \in M\}\,. \tag{40}$$

This implies $u = s \circ \bar{x} \in M$ for all $s \in U$ and we can define the function $f : U \to \mathbb{R}$ by

$$s \mapsto f(s) = f_{\varrho,X}(s \circ \bar{x}), \tag{41}$$

which is homogeneous of degree one and twice continuously differentiable on U. For all $u \in M$, the right-hand sides of (29) and (5), each in connection with the vector x defined by (4), coincide. To prove this, we show that the entries of underlying matrices R coincide. We start with the definition in (27):

$$\frac{1}{2}\frac{\partial^2}{\partial s_k \partial s_\ell} f^2(s)\Big|_{s=x} \overset{(41)}{=} \frac{1}{2}\frac{\partial^2}{\partial s_k \partial s_\ell} f^2_{\varrho,X}(s \circ \bar{x})\Big|_{s=x} = \frac{1}{2x_k x_\ell}\frac{\partial^2}{\partial u_k \partial u_\ell} f^2_{\varrho,X}\left(\mathbb{1}_n\right) \tag{42}$$

Moreover, for all $u \in M$, the function $f_x(u)$ defined in (28) in connection with the vector x defined by (4) coincides with $f_{\varrho,X}(u)$:

$$f_x(u) \overset{(28)}{=} f(u \circ x) \overset{(41)}{=} f_{\varrho,X}(u \circ x \circ \bar{x}) = f_{\varrho,X}(u) \tag{43}$$

Theorem 1 thereby implies all statements of Proposition 1.

D Calculating VaR sensitivities without Monte-Carlo simulations

This section outlines two situations in which the distribution function of the linear combination $X(u) = \sum_{i=1}^n u_i X_i$ has an analytical representation. The Value-at-Risk,

$$\mathrm{VaR}_{1-\alpha}\left(\sum_{i=1}^n u_i X_i\right) = F_{X(u)}^{-1}(1-\alpha) \tag{44}$$

can be determined numerically by the Newton method. First and second-order derivatives of $\mathrm{VaR}_{1-\alpha}\left(\sum_{i=1}^n u_i X_i\right)$ with respect to scalars u_i can also be calculated numerically.

In the first situation, we assume that the X_i, $i = 1, ..., n$ are independent and gamma distributed with shape parameter γ_i and rate parameter ϑ_i. If all rate parameters are equal, $\vartheta_1 = ... = \vartheta_n =: \vartheta$, the aggregate loss X is gamma distributed with shape parameter $\gamma_1 + ... + \gamma_n$ and rate parameter ϑ. For the case that the rate parameters are not all the same, Moschopoulos (1985, p. 543) provides an analytical representation of the distribution function of X, which we denote by

$$F_X(x) = F_{\Gamma^+}(x; \gamma_1, ..., \gamma_n, \vartheta_1, ..., \vartheta_n) \tag{45}$$

To determine the Value-at-Risk of a linear combination $\sum_{i=1}^{n} u_i X_i$, note that for a scalar $u_i > 0$ the product $u_i \cdot X_i$ is gamma distributed with shape parameter γ_i and rate parameter ϑ_i/u_i. Hence, the distribution function of the linear combination $u_1X_1 + ... + u_nX_n$ is given by

$$F_{\Gamma^+}(x; \gamma_1, ..., \gamma_n, \vartheta_1/u_1, ..., \vartheta_n/u_n) \tag{46}$$

The second situation generalizes the first one. Furman et al. (2020) propose the class of multivariate mixed-gamma distributions, which is flexible in terms of the shape of the univariate distributions and the stochastic dependencies between them. According to Furman et al. (2020, p. 8 f.), the n-dimensional mixed-gamma distribution is defined as follows: let $\boldsymbol{\kappa} = (\kappa_1, ..., \kappa_n)$ be a vector of discrete random variables which can assume non-negative integer values, and let $p_{\boldsymbol{\kappa}}(\boldsymbol{k}) = \mathbb{P}(\kappa_1 = k_1, ..., \kappa_n = k_n)$ denote the probability mass function of $\boldsymbol{\kappa}$ with $\boldsymbol{k} = (k_1, ..., k_n) \in \mathbb{N}_0^n$. Let $f_\Gamma(x; \gamma, \vartheta)$ denote the density function of the univariate gamma distribution with shape parameter γ and rate parameter ϑ. The random vector $\boldsymbol{\Gamma}^{(\boldsymbol{\kappa})} = (\Gamma_1^{(\kappa_1)}, ..., \Gamma_n^{(\kappa_n)})$ is distributed n-variate mixed-gamma if its density function is given by

$$f_{\boldsymbol{\Gamma}^{(\boldsymbol{\kappa})}}(x_1, ..., x_n) = \sum_{\boldsymbol{k} \in \mathbb{N}_0^n} p_{\boldsymbol{\kappa}}(\boldsymbol{k}) \prod_{i=1}^{n} f_\Gamma(x_i; \gamma_{k_i}, \vartheta_i) \tag{47}$$

where the shape parameters are determined by $\gamma_{k_i} = \gamma_i + k_i$ with $\gamma_i > 0$. Recall that $F_{\Gamma^+}(x; \gamma_1, ..., \gamma_n, \vartheta_1, ..., \vartheta_n)$ in (45) is the distribution function of the sum of independent gamma distributed random variables with different shape and rate parameters. Assuming that there is only a finite number of vectors $\boldsymbol{k}$ with positive probability $p_{\boldsymbol{\kappa}}(\boldsymbol{k})$, the distribution function of $X = \Gamma_1^{(\kappa_1)} + ... + \Gamma_n^{(\kappa_1)}$ is given by

$$\begin{aligned}
F_X(x) &= \int_{\{y\in\mathbb{R}_+^n \text{ such that } \sum_{i=1}^n y_i \le x\}} \cdots \int \sum_{\boldsymbol{k}\in\mathbb{N}_0^n} p_{\boldsymbol{\kappa}}(\boldsymbol{k}) \prod_{i=1}^n f_\Gamma(y_i;\gamma_{k_i},\vartheta_i)\mathrm{d}y_1....\mathrm{d}y_n \\
&= \sum_{\boldsymbol{k}\in\mathbb{N}_0^n} p_{\boldsymbol{\kappa}}(\boldsymbol{k}) \int_{\{y\in\mathbb{R}_+^n \text{ such that } \sum_{i=1}^n y_i \le x\}} \cdots \int \prod_{i=1}^n f_\Gamma(y_i;\gamma_{k_i},\vartheta_i)\mathrm{d}y_1....\mathrm{d}y_n \\
&= \sum_{\boldsymbol{k}\in\mathbb{N}_0^n} p_{\boldsymbol{\kappa}}(\boldsymbol{k}) F_{\Gamma^+}(x;\gamma_{k_1},...,\gamma_{k_n},\vartheta_1,...,\vartheta_n),
\end{aligned}$$

and the distribution function of the linear combination $u_1\Gamma_1^{(\kappa_1)} + ... + u_n\Gamma_n^{(\kappa_1)}$ is given by

$$\sum_{\boldsymbol{k}\in\mathbb{N}_0^n} p_{\boldsymbol{\kappa}}(\boldsymbol{k}) F_{\Gamma^+}(x;\gamma_{k_1},...,\gamma_{k_n},\vartheta_1/u_1,...,\vartheta_n/u_n)$$

E Derivatives of the function $g(u)$

Let $n \in \mathbb{N}$, R be a symmetrix matrix and $x \in \mathbb{R}^n$ such that $x^{\mathrm{T}}Rx > 0$. We consider the function

$$g : u = (u_1, \ldots, u_n)^{\mathrm{T}} \mapsto g(u) = \sqrt{\sum_{i,j=1}^n \varrho_{ij} u_i x_i u_j x_j} = \sqrt{(u \circ x)^{\mathrm{T}} R \, (u \circ x)}$$

The first-order partial derivative of g with respect to an entry u_k of u is obtained as

$$\frac{\partial}{\partial u_k} g(u) = \frac{\sum_{i=1}^n \varrho_{ki} u_i x_i}{\sqrt{(u \circ x)^{\mathrm{T}} R \, (u \circ x)}} \cdot x_k \tag{48}$$

In matrix notation and at $u = \mathbb{1}_n$, the gradient of g is determined as

$$\nabla_u \, g\,(\mathbb{1}_n) = \frac{(Rx) \circ x}{\sqrt{x^{\mathrm{T}} R \, x}} \tag{49}$$

The second-order partial derivatives of g with respect to entries u_k and u_ℓ of u are

$$\frac{\partial^2}{\partial u_k \partial u_\ell} g(u) = \left\{ \frac{\varrho_{k\ell}}{\sqrt{(u \circ x)^{\mathrm{T}} R (u \circ x)}} - \frac{\left(\sum_{i=1}^n \varrho_{ki} u_i x_i\right)\left(\sum_{j=1}^n \varrho_{\ell j} u_j x_j\right)}{\left\{(u \circ x)^{\mathrm{T}} R (u \circ x)\right\}^{3/2}} \right\} x_k x_\ell$$

and hence

$$\frac{\partial^2}{\partial u_k \partial u_\ell} g(\mathbb{1}_n) = \frac{\varrho_{k\ell} x_k x_\ell - \left.\left(\frac{\partial g}{\partial u_k} \frac{\partial g}{\partial u_\ell}\right)\right|_{u=\mathbb{1}_n}}{\sqrt{x^{\mathrm{T}} R x}}$$

Note that, as we have already observed in formula (39),

$$\frac{\frac{\partial^2}{\partial u_k \partial u_\ell} g^2(\mathbb{1}_n)}{2 x_k x_\ell} = \varrho_{k\ell}$$

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