Identifying Scenarios for the Own Risk and Solvency Assessment of Insurance Companies*

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Abstract

Most insurers in the European Union determine their regulatory capital requirements based on the standard formula of Solvency II. However, there is evidence that the standard formula inaccurately reflects insurers’ risk situation and may provide misleading steering incentives. In the second pillar, Solvency II requires insurers to perform a so-called “Own Risk and Solvency Assessment” (ORSA). In their ORSA, insurers must establish their own risk measurement approaches, including those based on scenarios, in order to derive suitable risk assessments and address shortcomings of the standard formula. The idea of this paper is to identify scenarios in such a way that the standard formula in connection with the ORSA provides a reliable basis for risk management decisions. Using an innovative method for scenario identification, our approach allows for a simple but relatively precise assessment of marginal and even non-marginal portfolio changes. We numerically evaluate the proposed approach in the context of market risk employing an internal model from the academic literature and the Solvency Capital Requirement (SCR) calculation under Solvency II.

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Keywords: Risk measurement, Enterprise Risk Management, Own Risk and Solvency Assessment, Solvency II

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1 Introduction

Even in modern regulation systems such as Solvency II, the majority of insurance companies determine their capital requirements using a pre-specified standard formula rather than a self-developed internal risk capital model. However, various academic papers show that the risk landscape may not be realistically depicted on the basis of such a standardized approach, which can lead to incorrect steering incentives when portfolio decisions are made. A relatively well known deficiency of the Solvency II standard formula is that it disregards credit risks of any government counterparties in the European Economic Area (EEA) or the Organisation for Economic Co-operation and Development (OECD). Gatzert and Martin (2012), for instance, identify this problem and show that there is a gap between the results of an internal risk model and those of the standard formula in the context of market risks. Becker and Ivashina (2015) find empirical evidence that regulatory constraints on insurance companies investing in the corporate bond market can lead to portfolio distortions. Chen et al. (2019) find that relying on the square root rule for the calculation of risk-based capital (RBC) may provide a misleading view of changing diversification effects. In the case of equity risk, Fischer and Schlütter (2015) show that insurers’ asset selection is highly sensitive to parameters of the standard formula and can end up too risky or conservative. Braun et al. (2017) find that the standard formula is likely to have an adverse impact on life insurers’ asset allocation, as it hinders insurers in selecting an efficient investment portfolio. Furthermore, Pfeifer and Strassburger (2008) provide evidence that for several classes of distributions, the standard formula underestimates the true Solvency Capital Requirements (SCR). A more general criticism regarding the regulatory framework is formulated by Scherer and Stahl (2021), stating that the Solvency II standard formula ”lacks sound economic and mathematical reasoning”. A mitigation of the standard formula’s deficiencies could potentially be offered by second pillar requirements. These requirements aim to improve insurers’ Enterprise Risk Management and to this end require — in addition to the standard formula — stress tests and scenario analyses. These approaches aim to extend the regulatory risk mea-

\footnote{For instance, the European Insurance and Occupational Pensions Authority (EIOPA) states in their annual Insurance Statistics report regarding the own funds, cf. EIOPA (2019), that 2,470 of the 2,658 insurers evaluated base their risk calculations on the standard formula provided by Solvency II.}

\footnote{Cf. BaFin (2016)}

\footnote{For instance, Schlütter (2021) identifies deficiencies in the measurement of interest rate risk under the Solvency II regulation and derives correlated scenarios to find an appropriate adjustment.}
surement “in order to provide an adequate basis for the assessment of the overall solvency needs”, cf. EIOPA (2015) Guideline 7. Similarly, the requirements for the Own Risk and Solvency Assessment (ORSA) of an insurer expect to “also take into account risks that are not or not adequately included in the standard formula, and must develop a suitable assessment procedure for them”, cf. BaFin (2016).

In general, stress scenarios are a tool to “help decision makers understand better the level of resilience of the organization”, cf. Albrecher et al. (2018, Chapter 5.5). In the literature, there are several different suggestions on how to identify (reverse) stress scenarios by stressing the underlying distribution of risk drivers: for example, Korn and Müller (2021) apply the worst-case scenario in the setting of a portfolio optimization. Pesenti et al. (2019) employ the Kullback-Leibler divergence measure, Makam et al. (2021) use the $\chi^2$ divergence considering a discrete sample to derive stress scenarios. Breuer and Csiszár (2013) suggest identifying stress scenarios by using a relative entropy measure to ensure plausibility. However, they do not address the ORSA requirements explicitly to quantify scenarios that are not captured by the standard formula. A different idea more in line with the ORSA requirements is provided by McNeil and Smith (2012), proposing the so-called “least solvent likely event” (LSLE), a deterministic scenario that allows for a reliable evaluation of risk resulting from a given portfolio.\footnote{Notably, McNeil and Smith (2012, Corollary 4.4) show that their suggestion coincides with the so-called ”gradient scenario”, which is often also referred to as Euler allocation, cf. for instance Tasche (2008).}

This paper proposes a new approach to identify ORSA scenarios. We assume that the insurer’s strategy can be expressed by an exposure vector $u \in \mathbb{R}^n$. The entries of $u$ can represent, for example, investments in asset categories or sizes of the insurer’s lines of business. Corresponding to strategy $u$, the function $f^{\text{true}}(u)$ provides the capital requirement which satisfies the safety level defined in pillar 1 (i.e. the 99.5% Value-at-Risk). $f^{\text{true}}(u)$ can be considered as the outcome of a perfect internal model. Function $f^{\text{SF}}(u)$ presents the capital requirement according to the standard formula. Our target is to identify scenarios which approximate the residual

$$f^{\text{true}}(u) - f^{\text{SF}}(u)$$
To this end, we consider $g_m(u)$ as the risk measurement of strategy $u$ based on $m$ scenarios. If $g_m(u)$ approximates the residual in (1), then the standard formula in connection with ORSA scenarios provides an approximation of the true risk:

$$f_{true}(u) \approx f_{SF}(u) + g_m(u)$$

Our technical basis to derive scenarios for function $g_m(u)$ are so-called "orthogonal convexity scenarios" (OCS) as proposed by Aigner and Schlüter (2022). The aim of OCS is to translate the risk measurement of a portfolio into a small number of multivariate realization vectors. In contrast to the aforementioned stress scenario literature, OCS therefore do not change the risk distribution. By construction, OCS are orthogonal in the sense of sensitivity-implied tail correlations, as proposed by Paulusch and Schlüter (2022), and hence their use is not limited to elliptical distributions. Additionally, by employing deterministic scenarios, a combination with the standard formula leads to a deterministic risk measurement in the right–hand side of Eq. (2). We show that such a combination with only a single OCS can then be set to reflect the outcome of the internal model for the initial portfolio as well as all first-order derivatives. The approach thus coincides with the well known Euler capital allocation. When additional scenarios are taken into account, second-order derivatives are also reflected correctly. By approximating the internal model not only in a linear way, the scenarios reflect how portfolio-wide diversification effects alter when the portfolio volumes are changed. Deriving scenarios for a representative insurer could thereby provide a regulatory authority with a tool that can be handed out to insurance companies as an addition to the standard formula.

Numerically, the suggestion is evaluated based on Gatzert and Martin (2012) and Eckert et al. (2016) dealing with market risks. In the latter, the authors employ an internal model which comprises the risks of three sub-modules, the outcome of which is considered as the true portfolio risk. A difference in risk capital between the internal model and the standard formula is found, providing a good starting point for the scenario-based extension of the regulatory approach.

This article contributes to the literature in three ways: Firstly, it highlights the shortcomings of a standardized approach to risk measurement. Secondly, a practical idea is presented on how to formalize and implement the suggestions of the ORSA. Thirdly, empirical indications are given as to what meaningful scenarios in the sense of the ORSA
might look like for an example company.

The remainder of this article is structured as follows. Section 2 presents the ideas of Aigner and Schlüter (2022) for the determination of scenarios. Section 2 summarizes the suggestion of Solvency II for the measurement of market risk as well as an internal model counterpart given by Eckert et al. (2016) and Gatzert and Martin (2012). Section 3 numerically evaluates the goodness of the suggested extension of the standard formula. Section 4 provides concluding remarks about the concrete practical usefulness of the proposed approach.

2 Orthogonal convexity scenarios for the ORSA

We suppose that a company’s risk can be specified by inspecting a random vector

\[ X = (X_1, \ldots, X_n)^T \]

with \( X_i \) modeling the losses (or gains in case of negative values) of the \( i \)th risk driver. The \( X_i \) could be losses resulting from risks of the various submodules in the standard formula, but they could also be defined at a more granular level. For example, in section 2 we will consider the \( X_i \) to reflect losses from single equity and bond investments. It is further assumed that the insurer can change its portfolio by linearly scaling the \( X_i \). To this end, we introduce an exposure vector \( u \in \mathbb{R}^n \) representing the portfolio volumes. The true risk function can then be defined, in line with Solvency II, as

\[ f^{true}: \mathbb{R}^n \to \mathbb{R} \]

\[ u \mapsto \varrho(X^T u) - \mathbb{E}(X^T u) \]

with \( \varrho \) being a risk measure. For the identification of scenarios later on, it is sufficient to assume that \( \varrho \) is homogeneous of degree one and law-invariant. Paulusch (2017) ensures that the common risk measure under Solvency II, the 99.5% Value-at-Risk, fulfills this property. Notably, our method could also be applied in connection with the Expected Shortfall, for example, cf. also Paulusch (2017). The function \( f^{true} \) could then be derived from a stochastic model or a “perfect” internal model.
Secondly, we will inspect the regulatory standard formula that measures the risk of portfolio $u$ in terms of

$$f^{SF}: \mathbb{R}^n \to \mathbb{R}, u \mapsto f^{SF}(u)$$

which is explicitly specified in the ORSA, cf. [EIOPA (2015)]\(^6\). We assume that both functions, $f^{true}(u)$ and $f^{SF}(u)$, are twice continuously differentiable in a neighborhood of an initial portfolio $u_{initial}$. The central concern that we would like to tackle is that $f^{true}(u)$ and $f^{SF}(u)$ may deviate, and we will thus inspect and approximate the residual

$$f^{diff}(u) = f^{true}(u) - f^{SF}(u)$$

Here, situations that are not captured by the standard formula can be considered. In order to approximate $f^{diff}(u)$ relying on deterministic scenarios also taking into account tail dependencies, the approach presented by [Aigner and Schlüter (2022)]\(^7\) will be used. The authors suggest employing so-called “orthogonal convexity scenarios” (OCS) resulting in a scenario-based risk-measurement function

$$g_m : \mathbb{R}^n \to \mathbb{R}$$

$$(u \mapsto \sqrt{\sum_{i=1}^{m} \left( (x_i^{OCS})^T u \right)^2})$$

with $1 \leq m \leq n$ a pre-specified number of scenarios that should be considered and $x_i^{OCS}$, $i = 1, \ldots, m$, denoting the OCS\(^8\). The latter are defined as

$$x_i^{OCS} = \frac{w_i^T H w_i}{\sqrt{2 \cdot w_i^T H w_i}}$$

with weightvectors $w_1, \ldots, w_m \in \mathbb{R}^n$ and $H$ the Hessian matrix of $(f^{diff})^2$\(^9\). The weightvectors have to be specified by the user of the method and are supposed to be selected orthogonally in the sense of the bilinearform

$$\langle w_i, w_j \rangle_H = w_i^T H w_j$$

\(^6\)Although this paper focuses on the standard formula for insurance companies, the methodology could also be adapted to other models which are to be compared to a benchmark model.

\(^7\)There is some literature dealing with tail dependencies such as [Mittnik (2014)] and [Paulusch and Schlüter (2022)].

\(^8\)These scenarios can be understood as realizations of the risk vector $X$ in Eq. (9).

\(^9\)Appendix A.2 highlights how to calibrate the Hessian matrix.
with $H$ as before. Appendix [A.3] provides some guidance on how to identify the necessary weightvectors. Explicitly, it can be ensured that

$$x_1^{OCS} = \nabla_u f^{\text{diff}}(u_{\text{initial}})$$

such that the first scenario coincides with the Euler allocation of $f^{\text{diff}}$. The OCS approach can also be interpreted as an extension of the "Least solvent likely event" (LSLE) introduced by [McNeil and Smith (2012)] by including additional scenarios that can capture the convexity of the approximated risk measurement.

By basing the identification of the scenarios on the Hessian matrix, the function $g_m$ is capable of capturing non-linear dependencies and heavy tails in the portfolio as outlined by [Paulusch and Schlüetter (2022)]. When now approximating $f^{\text{true}}(u)$ by the sum $f^{\text{SF}}(u) + g_m(u)$, it fulfills the properties summarized in Proposition 1.

**Proposition 1.** Let $1 \leq m \leq n$, $f^{\text{true}}(u)$ and $f^{\text{SF}}(u)$ as before and $f^{\text{true}}(u_{\text{initial}}) > f^{\text{SF}}(u_{\text{initial}})$. Then, it holds:

1) $f^{\text{true}}(u_{\text{initial}}) = f^{\text{SF}}(u_{\text{initial}}) + g_m(u_{\text{initial}})$

2) For all $v \in \mathbb{R}^n$, it is

$$\frac{\partial}{\partial h} f^{\text{true}}(u_{\text{initial}} + hv) |_{h=0} = \frac{\partial}{\partial h} (f^{\text{SF}}(u_{\text{initial}} + hv) + g_m(u_{\text{initial}} + hv)) |_{h=0}$$

3) For $v_1, v_2 \in \text{span}\{w_1, \ldots, w_m\}$, it is

$$\frac{\partial^2}{\partial h_1 \partial h_2} f^{\text{true}}(u_{\text{initial}} + h_1 v_1 + h_2 v_2) |_{h_1=h_2=0} = \frac{\partial^2}{\partial h_1 \partial h_2} (f^{\text{SF}}(u_{\text{initial}} + h_1 v_1 + h_2 v_2) + g_m(u_{\text{initial}} + h_1 v_1 + h_2 v_2)) |_{h_1=h_2=0}$$

Therein, $g_m$ is as in Eq. (6) in connection with $x_i^{OCS}$’s as in (7).

The proof is presented in Appendix [A.1]. From Proposition 1, we see that an extension of the standard formula adding risk resulting from OCS indeed allows an approximation of the true risk measurement function in the sense of first and second order sensitivities.

To derive ORSA scenarios in the sense of section [2] we have to calibrate the two risk
measurements based on the true risk which will be represented by an internal model and the one based on the regulatory requirements. This is done in the following two subsections. Specifically, we restrict the analysis to a market risk setting including equity, interest rate and spread risk, and set up a specific portfolio.

2.1 Specification of $f^{\text{true}}$

To meet the requirements of the regulatory authority in Europe, we employ as risk measure $\varrho$ in (3) the VaR to a confidence level of 99.5% such that the true risk is given by

$$f^{\text{true}} : \mathbb{R}^n \to \mathbb{R}$$

$$u \mapsto \text{VaR}_{0.995} \left( X^T u \right) - \mathbb{E} \left( X^T u \right)$$

with a random risk vector

$$X = (X_1, \ldots, X_n)^T$$

(9)

consisting of $n = n_B + n_S$ risk drivers comprising $n_B \in \mathbb{N}$ bond and $n_S \in \mathbb{N}$ stock investments. Without loss of generality, we assume that $X_1, \ldots, X_{n_B}$ reflect the losses/gains resulting from bonds and $X_{n_B+1}, \ldots, X_{n_B+n_S}$ those from stock investments. For $f^{\text{true}}$ to be well-defined, we then have to specify $X$. For notational reasons, we rewrite the risk vector $X = (X^B, X^S)^T$ to represent losses/gains resulting from bonds and stocks separately after one year. In order to determine these stochastic vectors

$$X^B = (X^B_1, \ldots, X^B_{n_B})^T$$

$$X^S = (X^S_1, \ldots, X^S_{n_S})^T$$

we follow [Eckert et al. (2016)], who suggest modeling stock investments in combination with a reduced form credit risk model for defaultable bond exposure. Therefore, they let the stochastic default intensity (hazard rate) $h(t)$ follow a Vasicek (1977) process. The
resulting model in Eckert et al. (2016) can be written as

\[
dr(t) = \kappa \cdot (\theta - r(t)) \, dt + \zeta \, dW_r(t)
\]
\[
dh_1(t) = \chi_1 \cdot (o_1 - h_1(t)) \, dt + \Gamma_1 \, dW_{h_1}(t)
\]
\[
\ldots
\]
\[
dh_{n_B}(t) = \chi_{n_B} \cdot (o_{n_B} - h_{n_B}(t)) \, dt + \Gamma_{n_B} \, dW_{h_{n_B}}(t)
\]
\[
dS_1(t) = \mu_1 S_1(t) \, dt + \sigma_1 \cdot S_1(t) \, dW_{S_1}(t)
\]
\[
\ldots
\]
\[
dS_{n_S}(t) = \mu_{n_S} S_{n_S}(t) \, dt + \sigma_{n_S} \cdot S_{n_S}(t) \, dW_{S_{n_S}}(t)
\]

where \( W(t) = (W_r(t), W_{h_1}(t), \ldots, W_{h_{n_B}}, W_{S_1}, \ldots, W_{S_{n_S}}) \) is a standard Brownian motion with a symmetric correlation matrix \( R_{IM} \) implying that the valuation of market risk takes into account interest rate risk, credit risk, equity risk as well as dependencies between them. Herein, \( \kappa \) and \( \chi_i \) define the speeds of mean reversion, \( \theta \) and \( o_i \) the long-term means, and \( \zeta \) and \( \Gamma_i \) the volatilities of the processes for \( i = 1, \ldots, n_B \). Additionally, stock investments are assumed to follow geometric Brownian motions. Here, there are closed form solutions for the pricing of stock investments given by

\[
S_i(t) = S_i(0) \cdot \exp \left( \mu_s \cdot \frac{\sigma_i^2}{2} \cdot t + \sigma_i \cdot \sqrt{t} \, W_i \right)
\]

for \( i = n_B + 1, \ldots, n_B + n_S \), cf. for instance Gatzert and Martin (2012, Eq. (7)), modeling the stock value at time \( t \). Since we are interested in the stochastic losses of each stock investment, we set

\[
(X^S)_i := -(S_i(1) - S_i(0))
\]

for \( i = 1, \ldots, n_S \). Notably, we will set \( S_i(0) = 1 \) later on.

For the evaluation of bond investments, we have to take into account spread and interest rate risk. As our model states, we assume the interest rate process to follow a Vasicek (1977) process

\[
dr(t) = \kappa \cdot (\theta - r(t)) \, dt + \zeta \, dW_r(t)
\]

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11 For more details on the model, cf. Eckert et al. (2016).

12 The selection of geometric Brownian motions for modeling stocks is quite common in the literature, cf. for instance Islam and Nguyen (2020) or Graf and Korn (2020).

13 In Eq. (10), negative values are reported such that positive values represent losses later on.
allowing us to determine the price of a non-defaultable zero coupon bond with maturity $T$ as\textsuperscript{14}

\[ P(t, T) = \exp(-M_r(t, T) + 0.5 \cdot V_r^2(t, T)), \]

with

\[ M_r(t, T) = r(t) \cdot \frac{1 - \exp(-\kappa \cdot (T - t))}{\kappa} + \theta \cdot \left( (T - t) - \frac{1 - \exp(-\kappa \cdot (T - t))}{\kappa} \right) \]

\[ V_r^2(t, T) = \frac{\zeta^2}{\kappa^2} \left( (T - t) - 2 \cdot \frac{1 - \exp(-\kappa (T - t))}{\kappa} + \frac{1 - \exp(-2 \kappa (T - t))}{2 \kappa} \right) \]

Furthermore, in line with \cite{Eckert2016} we follow \cite{Duffie1999} to account for credit risk in the valuation of defaultable bonds. Therefore, default events for a given bond $i$ are modeled by a Cox process with a stochastic hazard rate $h_i(t)$. Moreover, the model of \cite{Duffie1999} allows us to take into account dependencies between credit spread and interest rate by using correlated Brownian motions $W_r(t), W_{h_i}(t), i = 1, \ldots, n_B$. Additionally, a recovery of the market value is assumed such that in the case of a default at time $\tau$, each bond pays a fraction of its value before the default

\[ \delta_R(\tau) \cdot P_{RMV}(\tau, T) \]

with $\delta_R(t)$ denoting the recovery rate\textsuperscript{15}$P_{RMV}(t, T) = \lim_{s \to \tau} P_{RMV}(s, T)$ and $P_{RMV}(t, T)$ the pre-default price at time $t < \tau$ of a recovery of market value (RMV) defaultable bond with maturity $T$. Then, the price of a defaultable bond is

\[ P_{RMV}(t, T) = E^Q \left( \exp \left( - \int_t^T (r(u) + s_i(u)) \, du \right) \right) \]

with the credit spread $s_i(t) = (1 + \delta_R(t)) \cdot h_i(t)$. Since the hazard rates are assumed to follow a \cite{Vasicek1977} process, the spread risks $s_i(t)$ follow —according to Itô’s Lemma— again a \cite{Vasicek1977} process given by

\[ ds_i(t) = \chi_i \cdot (\dot{h}_i - s_i(t)) \, dt + \dot{\Gamma}_i \, dW_{h_i} \]

\textsuperscript{14}These formulas are presented for example in \cite{Eckert2016} and \cite{Schonbucher2003}.

\textsuperscript{15}In general, the recovery rate could be stochastic, but for simplicity, we assume a constant recovery rate later on.
with \( \hat{o}_i = (1 - \delta_R) \cdot o_i \), \( \hat{\Gamma}_i = (1 - \delta_R) \Gamma_i \) and a constant recovery rate \( \delta_R(t) = \delta_R \). Then, Eckert et al. (2016) provide the following closed form solutions for the price of a RMV defaultable bond as 

\[
P_{i}^{RMV}(t, T) = P(t, T) \exp \left( -M_{s_i}(t, T) + 0.5 \cdot V_{s_i}^2(t, T) + C_i(t, T) \right),
\]

with

\[
M_{s_i}(t, T) = s_i(t) \cdot \frac{1 - \exp(-\chi_i(T - t))}{\chi_i} + \hat{o}_i \cdot \left( (T - t) - \frac{1 - \exp(-\chi_i(T - t))}{\chi_i} \right),
\]

\[
V_{s_i}^2(t, T) = \frac{\hat{\Gamma}_i^2}{\chi_i^2} \left( (T - t) - 2 \cdot \frac{1 - \exp(-\chi_i(T - t))}{\chi_i} + \frac{1 - \exp(-2\chi_i(T - t))}{2\chi_i} \right),
\]

and

\[
C_i(t, T) = \rho_{r,h_i} \cdot \frac{\chi_i \hat{\Gamma}_i}{\kappa \chi_i} \left( (T - t) - \frac{1 - \exp(-\kappa(T - t))}{\kappa} - \frac{1 - \exp(-\chi(T - t))}{\chi} \right) + \frac{1 - \exp(-(\kappa + \chi_i)(T - t))}{\kappa + \chi_i},
\]

where \( \rho_{r,h_i} \) reflects the correlation of the standard Brownian motions \( W_r(t) \) and \( W_{h_i}(t) \). The price of a defaultable bond \( i \) with hazard rate \( h_i \) and maturity \( T_i \) at time \( t \) is then derived as

\[
B_i(t) = \sum_{h=t+1}^{T_i} CF_i(h) \cdot P_{i}^{RMV}(t, h)
\]

with cash flows

\[
CF_i(t) = \begin{cases} 
  c_i(t) \cdot FV_i, & \text{if } (t < T_i) \land (\tau_i^B > t) \\
  (1 + c_i(t)) \cdot FV_i, & \text{if } (t = T_i) \land (\tau_i^B > t) \\
  \delta_R \cdot B_i(t - 1), & \text{if } t = \tau_i^B \\
  0, & \text{else}
\end{cases}
\]

(11)

depending on the time of default \( \tau_i^B \), coupon \( c_i(t) \) and time \( t \). Herein, the face values \( FV_i \) are scaled such that \( B_i(0) = 1 \) for all \( i = 1, \ldots, n_B \). Since we are again interested in potential losses/gains after one year, we set

\[
\left( X_i^B \right)_i = -(B_i(1) - B_i(0))
\]

(12)

with \( i = 1, \ldots, n_B \). In case of a mixed portfolio consisting of stocks and bonds, the market value risk vector is then

\[
X = \left( (X_i^B)^T, (X_i^S)^T \right)^T
\]

with \( X_B \) and \( X_S \) as in (12) and (10) respectively. Here, the risk measure function in (3) is well-defined. By selecting the face value as in (11), the cash flows are adjusted in a
way that ensures that \( (X^T u)_i \) represents a loss with a value of \( u_i \) at \( t = 0 \). The portfolio volumes can be steered by adjusting the \( u_i \).

### 2.2 Specification of \( f^{SF} \)

The Solvency II framework suggests a module structure making it necessary to calculate \emph{equity}, \emph{interest rate} and \emph{spread} risks separately on a sub-module basis, cf. Figure 1. The risks are then aggregated towards the market module by the so-called square-root formula denoted by \( f^{SF} \). In particular, we calculate three different values: \( Mkt_{eq} \), \( Mkt_{int} \) and \( Mkt_{sp} \) representing the capital requirement of the sub-modules respectively. Liquidity, concentration, property and currency risk will be excluded in the following analysis for the sake of simplicity. In this section, we mainly adopt the notation of Gatzert and Martin (2012). For notational reasons, we set the exposure vector as 

\[
\begin{align*}
    u &= \left( (u^B)^T, (u^S)^T \right)^T \in \mathbb{R}^n \\
    u^B &= (u^B_1, \ldots, u^B_{n_B})^T \in \mathbb{R}^{n_B} \\
    u^S &= (u^S_1, \ldots, u^S_{n_S})^T \in \mathbb{R}^{n_S}
\end{align*}
\]

distinguishing between exposures referring to bond investments, \( u^B \in \mathbb{R}^{n_B} \), and those referring to stocks, \( u^S \in \mathbb{R}^{n_S} \).

Figure 1: Structure for the SCR calculation in the style of Gatzert and Martin (2012); only sub-modules that are taken into account later on are presented.
2.2.1 Interest rate risk

First, the risk resulting from a change of the term structure is determined within the interest rate risk sub-module. For this purpose, we calculate the present value (PV) of all interest-rate-sensitive exposures—namely $u^B$—by discounting their cash flows (CF) using the risk-free interest rate structure $r_f(t)$ which is published monthly by EIOPA, cf. Table 7. Specifically, we have to calculate for $i = 1, \ldots, n_B$

$$PV_{int}^i = \sum_{t=1}^{T_i} \frac{CF_i(t)}{(1 + r_f(t))^t}$$ (13)

with $T_i$ the maturity and $CF_i(t)$ the cash flow of investment $i$ at time $t$. For face values $FV_i$ as discussed in the section before and a coupon payment $c_i(t)$ at time $t$, the cash flow for bond $i$ is given by

$$CF_i(t) = \begin{cases} c_i(t) \cdot FV_i, & \text{if } t < T_i \\ (1 + c_i(t)) \cdot FV_i, & \text{if } t = T_i \end{cases}$$ (14)

for $i = 1, \ldots, n_B$. The upward shocked present values are then calculated as

$$(PV_{int}^{up})_i = \sum_{t=1}^{T_i} \frac{CF_i(t)}{(1 + \max (r_f(t) \cdot (1 + s_{up}(t)), 0.01))^t}$$ (15)

where the maximum in the denominator is in line with European Commission (2015, Article 166) and ensures that there is at least a shock of one percent. Furthermore, for the downward shock we calculate

$$(PV_{int}^{down})_i = \sum_{t=1}^{T_i} \frac{CF_i(t)}{(1 + \max (r_f(t) \cdot (1 + s_{down}(t)), 0)))^t}$$ (16)

with the maximum in the denominator accounting for the current low-level interest rate environment. In both cases, it is again $i = 1, \ldots, n_B$. The shocks $s_{up}(t), s_{down}(t)$ are provided by European Commission (2015, Article 166 and 167) and shown in Table 7.

Interpreting the results in Eq. (13), (15) and (16) as vectors in $\mathbb{R}^{n_B}$, allows us to determine the overall risk of the sub-module interest rate as

$$Mkt_{int}(u^B) = \max \left( (PV_{int}^{up} - PV_{int}^{down})^T u^B, (PV_{int}^{up} - PV_{int}^{down})^T u^B \right)$$

depending on the part of the exposure vector reflecting bond investments.
2.2.2 Spread risk

Changes in the credit spread on exposures are considered in the rating-based risk sub-module of the Solvency II standard approach. The risk consists of three uncorrelated groups: the SCR of bonds \( Mkt^{bonds}_{sp} \), of securization positions \( Mkt^{securization}_{sp} \) and of credit derivatives \( Mkt^{cd}_{sp} \) which are then easily added up to the total risk of the sub-module spread

\[
Mkt_{sp} = Mkt^{bonds}_{sp} + Mkt^{securization}_{sp} + Mkt^{cd}_{sp}
\]

For simplicity, we will restrict our analysis only to bond assets, ignoring securization positions and credit derivatives. The SCR calculation for spread risk then takes into account the current value \( MV_{sp,i}(0) = u_i^B \) of bond \( i = 1, \ldots, n_B \). The stress referring to each bond depends on shocks that can be specified by including their ratings, which are publicly available, and their durations. In order to determine the latter, we rely on the [Macaulay (1938)] duration with a floor of one, as suggested by [European Commission (2015, Article 176)]. Given that there is only one coupon period per year, it can be determined as

\[
duration_i = \min \left( \frac{\sum_{t=1}^{T_i} t \cdot CF_i(t) \cdot (1 + r_f(t))^{-t}}{\sum_{t=1}^{T_i} CF_i(t) \cdot (1 + r_f(t))^{-t}} \cdot \frac{1}{1 + r_{YTM}}, 1 \right)
\]

for \( i = 1, \ldots, n_B \) with \( r_f \) and \( CF_i(t) \) the cash flow as before. Furthermore, we need to specify the yield to maturity \( r_{YTM} \) which is obtained by solving

\[
PV^{int}_{i} = \sum_{t=1}^{T_i} CF_i(t) \cdot (1 + r_{YTM})^{-t}
\]

where \( PV^{int}_{i} \) is calculated as in [13] and \( T_i \) as before. Given the rating and the duration of each bond, [European Commission (2015, Article 176)] further outlines how to specify the stresses \( stress_i \) for \( i = 1, \ldots, n_B \) explicitly, cf. Tables 8 and 9. Here, the SCR of the spread risk sub-module (in the simplified version only including bonds) is calculated as

\[
Mkt_{sp} \left( u^B \right) = \max \left( \sum_{i=1}^{n_B} u_i^B \cdot stress_i, 0 \right) = \max \left( stress^T u^B, 0 \right)
\]

\[^{16}\text{For the numerical calculation, there are several common approaches, such as the Newton-Raphson method which we will employ later on.}\]
It should be noted that for bonds issued by governments belonging to the EEA or the OECD, the stress, according to BaFin (2016), is always zero percent, and hence the \( Mkt_{sp} \) is also zero when the exposure is a respective bond.

2.2.3 Equity risk

In order to calculate the capital requirements resulting from equity risk, we first have to cluster the \( n_S \) stock assets within our portfolio in (9) into “global” and “other”. The class “global” comprises all exposures transacted in countries that are members of the European Economic Area (EEA) or the Organisation for Economic Co-operation and Development (OECD), cf. CEIOPS (2010) and European Commission (2015). Without loss of generality, we assume that the first \( k_{global} \in \mathbb{N} \) entries of \( u \) represent the exposure to “global” investments and the rest \( k_{other} = n_S - k_{global} \) the exposure to “other” investments. “Global” stocks are easily multiplied with \( \text{shock}_{global} = 0.3 \) and “other” investments are assumed to have a higher risk and therefore are assigned a shock of \( \text{shock}_{other} = 0.4 \). With these specifications, one can directly calculate the market values of both classes by summing up the market values at time \( t = 0 \), denoted by \( MV_{eq,i}(0) = u_i^S \), \( i = 1, \ldots, n_S \) for all assets in the respective class and multiplying it with the respective shock parameter

\[
Mkt_{eq, \text{global}}(u^S) = \max \left( 0.3 \cdot \sum_{i=1}^{k_{global}} u_i^S, 0 \right)
\]

\[
Mkt_{eq, \text{other}}(u^S) = \max \left( 0.4 \cdot \sum_{i=k_{global}+1}^{n_S} u_i^S, 0 \right)
\]

with \( k_{global} + k_{other} = n_S \). In order to aggregate the classes with respect to diversification effects, EIOPA (2011) recommends the aggregation via the square-root formula

\[
Mkt_{eq}(u^S) = \sqrt{x^T R_{eq} x}
\]

\[\text{17}\]The stresses are adjusted here to avoid pro-cyclical effects of adverse capital market developments, cf. Gatzert and Martin (2012) for more details.

\[\text{18}\]For further details on the specification of the shocks including strategic participation, adjustments etc. cf. Gatzert and Martin (2012). These are excluded in this paper for the sake of tractability, but the process in the simulation study later on would work equivalently.
with \( x = (\text{Mkt}_{\text{eq,global}}(u^S), \text{Mkt}_{\text{eq,other}}(u^S))^\top \) and the correlation matrix

\[
\begin{pmatrix}
    \text{Global} & \text{Other} \\
    1 & 0.75 \\
    0.75 & 1
\end{pmatrix}
\]

Notably, a portfolio only consisting of stocks is just exhibited to the equity sub-module, and the solvency capital requirement (SCR) calculation of the market risk module ends with this sub-module, cf. Figure 1.

### 2.2.4 Aggregation of the sub-modules

Following [European Commission (2015)](https://doi.org/10.1148/rg.2015040030), the three sub-modules, equity, interest rate and spread are assumed to be correlated by

\[
R_{\text{SF}} = \begin{pmatrix}
    1 & A & A \\
    A & 1 & 0.75 \\
    A & 0.75 & 1
\end{pmatrix}
\]

where the correlation parameter \( A \) is conditional on the result of \( \text{Mkt}_{\text{int}} \) as

\[
A = \begin{cases}
    0.5, & \text{if } \text{Mkt}_{\text{int}}(u^B) = (PV_{\text{int}} - PV_{\text{down}})^\top u^B \\
    0, & \text{if } \text{Mkt}_{\text{int}}(u^B) = (PV_{\text{int}} - PV_{\text{up}})^\top u^B
\end{cases}
\]

Then, the overall market risk of a portfolio \( u \in \mathbb{R}^{n_x+n_B} \) can be calculated with a square-root formula resulting in a specification of the function in (14) as

\[
f^{\text{SF}} : \mathbb{R}^n \to \mathbb{R}
\]

\[
u \mapsto \sqrt{x^\top R_{\text{SF}} x}
\]
with $x = (\text{Mkt}_{\text{int}}(u^B), \text{Mkt}_{\text{eq}}(u^S), \text{Mkt}_{\text{sp}}(u^B))^\top \in \mathbb{R}^n$ and $n = n_B + n_S$. It is shown in Appendix A.5 that $f^{\text{SF}}$ is indeed homogeneous of degree one, a necessary property for the application of OCS in the latter.

### 2.3 Portfolio set-up

Let us now construct a (theoretical) portfolio, the market risk of which will be determined on the basis of the approaches presented in the last two sections. We assume an investment portfolio consisting of $n_B = 5$ bond investments, cf. Table 1 and $n_S = 2$ stocks, cf. Table 2. Furthermore, we assume that we are equally invested in each of those positions such that we set the initial portfolio as $u_{\text{initial}} = \mathbf{1}_7$ – with $u_1, \ldots, u_5$ presenting the bond and $u_6, u_7$ the stock investments. Since there is a strict distinction between spread and default risk in the regulatory requirements, we assume that none of the bonds default until their maturity. We exclude default risk in the internal model here, such that the numerical analysis is indeed in line with the solvency calculation of the market risk module.

The SCR based on the standard formula can now directly be derived on the basis of the given specifications. Notably, the spread risk for the governmental bonds is set to zero, since Germany and Spain are both part of the EEA.

For the internal model, we take into account correlated standard Wiener processes with the correlation matrix $R_{IM}$ as discussed in section 2.1. The entries of $R_{IM}$ have been estimated relying on monthly data between 09/2011 and 09/2021.

<table>
<thead>
<tr>
<th>$B_i$</th>
<th>Type$_i$</th>
<th>rating$_i$</th>
<th>maturity$_i$</th>
<th>Coupon (in %)</th>
<th>$c_i$</th>
<th>$\chi_i$</th>
<th>$o_i$</th>
<th>$\Gamma_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Corporate</td>
<td>AA</td>
<td>16</td>
<td>8.00</td>
<td>0.0392</td>
<td>0.0269</td>
<td>0.0004</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>Corporate</td>
<td>A</td>
<td>12</td>
<td>2.95</td>
<td>0.0180</td>
<td>0.0240</td>
<td>0.0009</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>Corporate</td>
<td>BBB</td>
<td>11</td>
<td>5.75</td>
<td>0.0373</td>
<td>0.0453</td>
<td>0.0027</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>Government</td>
<td>BB</td>
<td>10</td>
<td>1.75</td>
<td>0.2201</td>
<td>0.5670</td>
<td>0.2299</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>Government</td>
<td>A</td>
<td>10</td>
<td>0.50</td>
<td>0.0139</td>
<td>-0.0070</td>
<td>0.0022</td>
<td></td>
</tr>
</tbody>
</table>

Table 1: Specifications of the bonds that are taken into account. $B_1$: Deutsche Bank AG, $B_2$: Commerzbank AG, $B_3$: E.ON SE, $B_4$: Greece and $B_5$: Spain. The parameters $\chi_i$, $o_i$ and $\Gamma_i$ refer to the Vasicek processes and are estimated on the basis of spread data between 09/2011 and 09/2021.

The choice of the initial portfolio is arbitrary thanks to homogeneity of $f^{\text{true}}$ and $f^{\text{SF}}$ as long as $f^{\text{true}}(u_{\text{initial}}) > f^{\text{SF}}(u_{\text{initial}})$. 

---

19 The choice of the initial portfolio is arbitrary thanks to homogeneity of $f^{\text{true}}$ and $f^{\text{SF}}$ as long as $f^{\text{true}}(u_{\text{initial}}) > f^{\text{SF}}(u_{\text{initial}})$.
Table 3: The stocks are then modeled by Monte Carlo simulations with 5,000,000 paths following geometric Brownian motions with the parameters as in Table 2. Here, \( S_1 \), namely Euro Stoxx, represents the investment in a “global” asset and \( S_2 \), Shanghai Stock Exchange (SSE) composite index, in an “other” asset. Both potential classes in the equity risk sub-module are thus covered.

The interest rate is modeled on the basis of a Vasicek (1977) process with a long–term

\[
\begin{array}{cccc}
S_i & \text{Index}_i & \text{category}_i & \mu_i & \sigma_i \\
1 & \text{Euro Stoxx} & \text{Global} & 0.0750 & 0.1611 \\
2 & \text{Shanghai SE composite index} & \text{Other} & 0.0632 & 0.2091 \\
\end{array}
\]

Table 2: Annualized parameters for the specification of the geometric Brownian motions representing stock investments. The values are estimated on monthly data between 09/2011 and 09/2021.

mean \( \theta = -0.0225 \), the speed of mean reversion of \( \kappa = 0.0046 \) and a drift of \( \sigma = 0.0015 \). The parameters have been estimated on the basis of monthly EURIBOR data between 09/2011 and 09/2021 and relying on Maximum Likelihood estimation, cf. for instance Fergusson and Platen (2015). Furthermore, the initial value is set to \( r(0) = \theta \).

As possible bond investments, a mixture of corporate and government bonds is considered. Their specifications are presented in Table 4. Employing again Maximum Likelihood estimation, the hazard processes can be fitted with the parameters presented in Table 1. Additionally, the recovery rate is set constant to \( \delta_R = 0.61 \).

With these specifications, we can now define

\[
\begin{array}{cccccccc}
r & h_1 & h_2 & h_3 & h_4 & h_5 & S_1 & S_2 \\
1 & 0.1864 & 0.1025 & 0.0848 & -0.0186 & 0.1617 & -0.0865 & -0.2118 \\
2 & 0.4518 & 0.2443 & 0.3165 & -0.0846 & 0.4263 & -0.0213 & 0.0629 \\
3 & 0.0848 & 0.2443 & 0.3165 & -0.0846 & 0.4263 & -0.0213 & 0.0629 \\
4 & -0.0186 & -0.0846 & -0.0147 & -0.0832 & -0.0832 & -0.0832 & -0.0832 \\
5 & 0.1617 & 0.4263 & 0.6409 & 0.5236 & 0.0789 & 0.0789 & 0.0789 \\
S_1 & -0.0865 & -0.0213 & -0.0916 & -0.2930 & -0.0095 & -0.0198 & -0.0198 \\
S_2 & -0.2118 & 0.0629 & -0.1576 & -0.1366 & 0.0789 & -0.1818 & 0.3365 \\
\end{array}
\]

Table 3: The entries of correlation matrix \( R_{IM} \). The values represent the correlation between stocks, bonds and interest rate \( r \) based on monthly data from 09/2011 to 09/2021.

\(^{20}\)Eckert et al. (2016) note a high sensitivity of the model to changes of \( \delta_R \), but for our purpose a change in the recovery rate would lead to similar results.
We can now identify OCS as in Eq. (7) such that \( g_m(u) \) as in Eq. (6) approximates \( f_{\text{diff}} \).

To this end, Appendix A.2 and Appendix A.3 provide the necessary technical details specifying the Hessian matrix \( H \) of \( (f_{\text{diff}})^2 \) and the weight vectors \( w_i, i = 1, \ldots, m \), which are crucial for the definition.

### 3 Results

Let us put ourselves in the situation of an investor who has seven units to invest in the portfolio specified in section 2.3. Table 4 shows what the risk capital looks like when investing all units separately in each asset and when investing in an equally weighted portfolio \( u_{\text{initial}} = \mathbb{1}_7 \). We observe that there is a severe gap between the two approaches when investing in assets separately. And even including diversification effects, we observe a severe underestimation of the true risk by

\[
\frac{f_{\text{SF}}(u_{\text{initial}})}{f_{\text{true}}(u_{\text{initial}})} - 1 = \frac{1.300}{1.478} - 1 = -12.04\%
\]

Such an underestimation may lead to a capital buffer too low to cover investment risks, as outlined for instance by Asadi and Al Janabi (2020).

Inspecting a common tool of capital allocation, the Euler allocation, of both measurements at \( u_{\text{initial}} \), cf. Table 5, also provides evidence that there is a severe gap between the two approaches. Notably, the entries of the gradient add up to the capital requirement for \( u_{\text{initial}} \) in both cases, cf. Tasche (2008). We can thus conclude that changing...
Table 5: Euler allocations of \( f^{\text{true}} \) and \( f^{\text{SF}} \) and their relative difference. \( \nabla_u \) represents the gradient.

<table>
<thead>
<tr>
<th>i</th>
<th>( \nabla_u f^{\text{SF}}_i )</th>
<th>( \nabla_u f^{\text{true}}_i )</th>
<th>Relative difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.126</td>
<td>0.068</td>
<td>85.29%</td>
</tr>
<tr>
<td>2</td>
<td>0.136</td>
<td>0.062</td>
<td>122.95%</td>
</tr>
<tr>
<td>3</td>
<td>0.190</td>
<td>0.058</td>
<td>229.34%</td>
</tr>
<tr>
<td>4</td>
<td>0.316</td>
<td>0.756</td>
<td>-58.20%</td>
</tr>
<tr>
<td>5</td>
<td>0.056</td>
<td>0.064</td>
<td>-11.11%</td>
</tr>
<tr>
<td>6</td>
<td>0.201</td>
<td>0.203</td>
<td>-1.96%</td>
</tr>
<tr>
<td>7</td>
<td>0.277</td>
<td>0.267</td>
<td>3.74%</td>
</tr>
<tr>
<td>( \sum )</td>
<td>1.300</td>
<td>1.478</td>
<td>-12.04%</td>
</tr>
</tbody>
</table>

diversification effects are not captured when basing risk measurement on the square-root formula, since the slope of the two functions in \( u_{\text{initial}} \) strongly differ. This observation is in line with Chen et al. (2019), who find empirical evidence that the standard formula does not reflect changing diversification effects correctly.

To approximate then the difference \( f^{\text{diff}}(u) = f^{\text{true}}(u) - f^{\text{SF}}(u) \) the OCS provided in Table 6 are determined as suggested in section 2 such that we obtain an approximation \( f^{\text{SF}}(u) + g_m(u) \) of \( f^{\text{true}}(u) \) in the sense of Proposition 1. In a first step, we use only a single scenario. That scenario \( x_i^{\text{OCS}} \) mainly reflects losses in bond investment \( B_4 \) and is equal to the Euler allocation of \( f^{\text{diff}} \) such that

\[
\sum_{i=1}^{7} x_i^{\text{OCS}} = 0.178 = f^{\text{diff}}(u_{\text{initial}}) = f^{\text{true}}(u_{\text{initial}}) - f^{\text{SF}}(u_{\text{initial}})
\]

Table 6: Orthogonal convexity scenarios for the definition of the approximation function \( g_m(u) \) as in section 2.

<table>
<thead>
<tr>
<th>i</th>
<th>Asset</th>
<th>( x_1^{\text{OCS}} )</th>
<th>( x_2^{\text{OCS}} )</th>
<th>( x_3^{\text{OCS}} )</th>
<th>( x_4^{\text{OCS}} )</th>
<th>( x_5^{\text{OCS}} )</th>
<th>( x_6^{\text{OCS}} )</th>
<th>( x_7^{\text{OCS}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( B_1 )</td>
<td>-0.057</td>
<td>-0.001</td>
<td>0.027</td>
<td>-0.034</td>
<td>0.013</td>
<td>0.012</td>
<td>0.004</td>
</tr>
<tr>
<td>2</td>
<td>( B_2 )</td>
<td>-0.075</td>
<td>-0.001</td>
<td>0.026</td>
<td>-0.031</td>
<td>0.011</td>
<td>0.008</td>
<td>0.002</td>
</tr>
<tr>
<td>3</td>
<td>( B_3 )</td>
<td>-0.130</td>
<td>0.002</td>
<td>0.044</td>
<td>-0.016</td>
<td>-0.011</td>
<td>-0.008</td>
<td>-0.002</td>
</tr>
<tr>
<td>4</td>
<td>( B_4 )</td>
<td>0.440</td>
<td>-0.002</td>
<td>-0.020</td>
<td>0.113</td>
<td>-0.088</td>
<td>-0.003</td>
<td>-0.016</td>
</tr>
<tr>
<td>5</td>
<td>( B_5 )</td>
<td>0.006</td>
<td>-0.002</td>
<td>-0.004</td>
<td>-0.048</td>
<td>0.040</td>
<td>-0.006</td>
<td>0.004</td>
</tr>
<tr>
<td>6</td>
<td>( S_1 )</td>
<td>0.005</td>
<td>0.071</td>
<td>-0.038</td>
<td>0.007</td>
<td>0.006</td>
<td>-0.005</td>
<td>0.071</td>
</tr>
<tr>
<td>7</td>
<td>( S_2 )</td>
<td>-0.011</td>
<td>-0.069</td>
<td>-0.034</td>
<td>0.008</td>
<td>0.029</td>
<td>0.002</td>
<td>-0.063</td>
</tr>
</tbody>
</table>

\[ ^{21} \text{Notably, for the determination it has been set } w_1 = u_{\text{initial}}, w_2 = (0, 0, 0, 0, 0, 1, -1)^T \text{ and } w_3 = (0, 2, 0, 2, 2, 0, 2, 0, 5, -0.5)^T \text{ since those are the business decisions we want to evaluate later on. Appendix A.3 sketches how Aigner and Schlätter (2022) select the OCS without pre-given decisions considered as well, and the following calculations could be conducted in the same way.} \]
Including this additional scenario comes with two advantages: Firstly, we ensure that the risk resulting from the initial portfolio \( u_{\text{initial}} \) is estimated precisely. Secondly, first order sensitivities of the true risk are met by our approximation. Employing further scenarios then allows us to fit a quadratic approximation of \( f^{\text{true}} \) in \( u_{\text{initial}} \) even allowing us to meet second order sensitivities of the true risk landscape.

For illustration, let us numerically evaluate the goodness of the approximation. Therefore, we shift the portfolio in the direction

\[
u_{\text{new}}(h) = u_{\text{initial}} + h \cdot (0, 0, 0, 0, 0, 1, -1)^T
\]

for \( h \in \mathbb{R} \). This new portfolio represents a shift between stock investments keeping the exposure to the five bond investments constant. For positive values of \( h \), we shift our portfolio from the “other” investment, \( S_2 \), in the direction of the “global” one, \( S_1 \). For negative \( h \) the shift is opposite, from “global” to “other”.

The resulting SCRs based on the different approaches are presented in Figure 2 for \( h \in [-1, 1] \). There, we see that the overall capital requirement according to the standard formula (red curve) always underestimates the true risk (black curve). Furthermore, we see that it strictly decreases when shifting the portfolio away from \( S_2 \) in the direction of \( S_1 \), which is reflected by the consistently negative slope. However, the true risk actually increases if the portfolio is shifted “too far” away from \( S_2 \) (positive \( h \)) due to changing diversification effects that are not captured by the standard formula. Generally, we can say that first and second order sensitivities in \( u_{\text{initial}} = 17 \) (for \( h = 0 \)) are not reflected correctly.

By extending the standard formula with only \( m = 1 \) OCS (orange curve), we can adjust the risk measurement function in a sense that the true risk in \( u_{\text{initial}} \) is reflected correctly and that even first order sensitivities of \( f^{\text{true}} \) in all directions are met. The latter is due to the selection of the Euler allocation of \( f^{\text{diff}} \) as first scenario. Such an extension is a good starting point, but also does not take into account changing diversification effects, since it still has a consistently negative slope in line with the standard formula.

Including \( m = 2 \) scenarios (blue curve) allows a fit with a quadratic approximation function that also overcomes this second problem. We observe that now—additional to the properties of the extension with one scenario—even second order sensitivities are met in a neighbourhood of \( u_{\text{initial}} \). We thus obtain a risk measurement function that is also
capable of evaluating non-marginal portfolio shifts.

In order to inspect the goodness of the extension more granularly, let us inspect a more complex shift impacting all portfolio positions simultaneously

$$u_{\text{new}}(h_1, h_2) = u_{\text{initial}} + h_1 \cdot \left( \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, -\frac{1}{2}, -\frac{1}{2} \right)^T + h_2 \cdot (0, 0, 0, 0, 0, 1)^T \quad (19)$$

for $h_1, h_2 \in \mathbb{R}$. If we set there $h_1 = 0$, we would again inspect a portfolio shift in the sense of Eq. $[18]$. Setting $h_2 = 0$ would then reflect a shift away from stock into bond investments for positive $h$ and vice versa for negative ones. Notably, the directions have been chosen such that the overall investment sum does not change. Comparing the resulting SCRs on the basis of OCS leads to the results presented in Figure 3 for $h_1, h_2 \in [-1, 1]$.

There, the left part shows the relative error of the standard formula extended by $m = 1$ scenario. It should be noted that we thereby obtain a reasonable approximation of $f_{\text{true}}$ if we only evaluate marginal portfolio changes. When $h_2$ is de-/increased too far, the underlying curvature of the true risk measurement function cannot be reflected any more. That misestimation is represented by the relative errors of $f^{SF}(u_{\text{new}}(h_1, h_2)) + g_1(u_{\text{new}}(h_1, h_2))$ (red parts in the figure).
By including a second scenario, we are still able to approximate the true risk in a neighborhood of \( u_{initial} \), but changing diversification effects are also considered. In the middle part of Figure 3, we observe that the gray parts—reflecting a relative error of about 0%—are not linear any more, but are instead spread in all directions. At the same time, we have to accept that a slight overestimation of the true risk may occur (blue parts), which is due to the fact that we extend the standard formula by a strictly positive function \( g \). Numerically, including a second and third scenario reduces the absolute amount of the relative error from 5.86% (left) to 3.39% (middle). Notably, the inclusion of the third scenario (right part) does not have a great impact on the maximal relative error (3.00%), but indeed the neighbourhood region in which the approximation meets the true risk (gray parts in the right) can be widened. The inclusion of even more scenarios here also seems reasonable.
4 Conclusion

This paper suggests “orthogonal convexity scenarios” (OCS) to address the requirement for the Own Risk and Solvency Assessment (ORSA) of an insurer employing scenario analyses. Explicitly, it is expected to “also take into account risks that are not or not adequately included in the standard formula, and [...] develop a suitable assessment procedure for them” (BaFin (2016)). We show that the OCS allow the derivation of a deterministic extension of the standard formula such that the overall risk is in line with the true portfolio risk. The approach is applied in the context of market risks comprising interest rate, spread and equity risks. For the set-up of the standard formula, we follow the regulatory requirements and the the approaches from the literature. Additionally, we use an internal model which is supposed to represent the true risk resulting from an asset portfolio. Notably, the true risk is generally unknown in practice, at which point an approximation becomes necessary. We find that extending the standard formula by OCS provides a reasonable approximation of the true risk in the sense of first and second order sensitivities. The latter property allows the evaluation of even non-marginal portfolio shifts, since changing diversification effects are considered by the resulting risk measurement. Although the approximation is only local, the examples in Paulusch and Schlüter (2022) indicate that the methodology is useful for other portfolios in addition to the calibration portfolio. The suggested approach can thus be seen as an answer to the question of how to select scenarios in the ORSA to measure risks that are not captured by the standard formula. Notably, since only deterministic scenarios are taken into account, we provide an easy tool for communicating the difference between an internal model and the standard formula to decision-makers, which has been identified as one of the fundamental aims of stress scenarios, cf. Albrecher et al. (2018, Chapter 5.5).
A  Appendix

A.1 Proof of Proposition 1

Let $u_{\text{initial}}$ be arbitrary and assume $f^{\text{true}}(u_{\text{initial}}) > f^{\text{SF}}(u_{\text{initial}})$. Furthermore, assume that $f^{\text{true}}$ and $f^{\text{SF}}$ are homogeneous of degree one and twice continuously differentiable in $u_{\text{initial}}$. Then

$$f^{\text{diff}}(u) := f^{\text{true}}(u) - f^{\text{SF}}(u)$$

directly fulfills these properties as well. According to Aigner and Schlütter (2022, Theorem 1) it then holds for the approximation function $g$ as specified in section 2

$$f^{\text{diff}}(u_{\text{initial}}) = g_m(u_{\text{initial}})$$

$$\frac{\partial}{\partial h} f^{\text{diff}}(u_{\text{initial}} + hv) \bigg|_{h=0} = \frac{\partial}{\partial h} (g_m(u_{\text{initial}} + hv) \bigg|_{h=0})$$

$$\frac{\partial^2}{\partial h_1 \partial h_2} f^{\text{diff}}(u_{\text{initial}} + h_1 v_1 + h_2 v_2) \bigg|_{h_1=h_2=0} = \frac{\partial^2}{\partial h_1 \partial h_2} g_m(u_{\text{initial}} + h_1 v_1 + h_2 v_2) \bigg|_{h_1=h_2=0}$$

for all $v \in \mathbb{R}^n$ and $v_1, v_2 \in \text{span}\{w_1, \ldots, w_m\}$. Thereby, Proposition 1 follows.

A.2 Estimation of the Hessian matrix

For the identification of OCS in Eq. (7), we have to estimate the Hessian matrix of $(f^{\text{diff}})^2$. Employing the chain rule leads to

$$H = 2 \cdot (H^{f^{\text{true}}} - H^{f^{\text{SF}}}) \cdot (f^{\text{diff}}(u_{\text{initial}})) + 2 \cdot (\nabla_u f^{\text{diff}}(u_{\text{initial}}))^T \cdot \nabla_u f^{\text{diff}}(u_{\text{initial}})$$

(20)

with $H^{f^{\text{true}}}$ and $H^{f^{\text{SF}}}$ the Hessian matrices of $f^{\text{SF}}$ and $f^{\text{true}}$ respectively, and $\nabla_u f^{\text{diff}}$ the gradient of $f^{\text{diff}}$ all evaluated at $u_{\text{initial}}$. Furthermore, the gradient simplifies to

$$\nabla_u f^{\text{diff}}(u_{\text{initial}}) = \nabla_u f^{\text{true}}(u_{\text{initial}}) - \nabla_u f^{\text{SF}}(u_{\text{initial}})$$

It is then necessary to estimate the single parts of Eq. (20). On the one hand, $H^{f^{\text{SF}}}$ and $\nabla_u f^{\text{SF}}(u_{\text{initial}})$ can be determined easily by numerical derivation providing a reasonable result, since $f^{\text{SF}}$ is deterministic. The identification of $H^{f^{\text{true}}}$ and $\nabla_u f^{\text{true}}(u_{\text{initial}})$, on the other hand, is more challenging. Monte Carlo simulations can be applied in line with Gourieroux et al. (2000) and Tasche (2009) who suggest Kernel estimators leading to a consistent estimation of $H^{f^{\text{true}}}$ and $\nabla_u f^{\text{true}}(u_{\text{initial}})$. Thanks to Slutsky’s Theorem, (cf.
Casella and Berger (2002, p. 239 f.), we can then consistently estimate $H$ in (20), since it is a composition of consistent estimations.

### A.3 Selection of weightvectors

For the identification of the necessary weightvectors $w_1, \ldots, w_m$ in Eq. (7), the user of the methodology could follow Appendix B in Aigner and Schlüter (2022) and set $w_1 = u_{\text{initial}}$. For the identification of $w_2, \ldots, w_m$, the authors define a matrix $M \in \mathbb{R}^{n \times \tilde{n}}$ with $\tilde{n} = \text{rank}(H)$, such that

$$(Mv)^T \cdot H \cdot (Mv) = 0$$

for all $v \in \mathbb{R}^{\tilde{n} - m}$ and the columns of $M$ denoted as $M_i$ also fulfill $(M_i)^T HM_j = 0$, for $i \neq j$. Further, they define the diagonal matrix $\Lambda = M^T HM$ and denote by $\Lambda^{-0.5}$ the diagonal matrix with entries $\lambda_1^{-0.5}, \ldots, \lambda_{\tilde{n} - m}^{-0.5}$. The weightvectors can then be determined as

$$w_{m+1} = \frac{1}{\sqrt{s^T \Lambda^{-0.5} M^T \Lambda^{-0.5} s}} \cdot M \Lambda^{-0.5} s$$

where $s$ is the eigenvector of $\Lambda^{-0.5} M^T M \Lambda^{-0.5}$ which refers to the smallest eigenvalue.
A.4 Parameters for regulatory capital requirement

<table>
<thead>
<tr>
<th>$t$</th>
<th>Risk-free interest rate $r_f(t)$</th>
<th>Relative change $s^{up}(t)$</th>
<th>Relative change $s^{down}(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.00612</td>
<td>0.7</td>
<td>-0.75</td>
</tr>
<tr>
<td>2</td>
<td>-0.00594</td>
<td>0.7</td>
<td>-0.65</td>
</tr>
<tr>
<td>3</td>
<td>-0.00556</td>
<td>0.64</td>
<td>-0.56</td>
</tr>
<tr>
<td>4</td>
<td>-0.00513</td>
<td>0.59</td>
<td>-0.5</td>
</tr>
<tr>
<td>5</td>
<td>-0.00462</td>
<td>0.55</td>
<td>-0.46</td>
</tr>
<tr>
<td>6</td>
<td>-0.00353</td>
<td>0.52</td>
<td>-0.42</td>
</tr>
<tr>
<td>7</td>
<td>-0.00293</td>
<td>0.49</td>
<td>-0.39</td>
</tr>
<tr>
<td>8</td>
<td>-0.00232</td>
<td>0.47</td>
<td>-0.36</td>
</tr>
<tr>
<td>9</td>
<td>-0.00172</td>
<td>0.44</td>
<td>-0.33</td>
</tr>
<tr>
<td>10</td>
<td>-0.00114</td>
<td>0.43</td>
<td>-0.31</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>

Table 7: Risk-free interest rate $r_f$ structure provided by EIOPA. The data are from 08/2021. Additionally, upward and downward shocks for the interest rate module are presented, cf. European Commission (2015).

<table>
<thead>
<tr>
<th>Duration ($dur_i$) in years</th>
<th>$stress_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$dur_i \leq 5$</td>
<td>$3% \cdot dur_i$</td>
</tr>
<tr>
<td>$5 &lt; dur_i \leq 10$</td>
<td>$15% + 1.7% \cdot (dur_i - 5)$</td>
</tr>
<tr>
<td>$10 &lt; dur_i \leq 20$</td>
<td>$23.5% + 1.2% \cdot (dur_i - 10)$</td>
</tr>
<tr>
<td>$dur_i &gt; 20$</td>
<td>$\min (35% + 0.5% \cdot (dur_i - 20), 1)$</td>
</tr>
</tbody>
</table>

Table 8: Parameters for the determination of $stress_i$ within the spread sub-module under Solvency II regulation for bonds unrated by nominated ECAI are reported.
Table 9: Parameters for the determination of $stress_i$ within the spread sub-module under Solvency II regulation for bonds rated by a nominated External Credit Assessment Institution (ECAI). All values for $a_i$ and $b_i$ are in percent.
A.5 Homogeneity of degree one of $f^{\text{SF}}$

We need to show that $f^{\text{SF}}$ in (17) is homogeneous of degree one. Therefore, we let every figure as in section 2.2. We can calculate for $\lambda \in \mathbb{R}$ and $u = \left( (u^B)^T, (u^S)^T \right)^T \in \mathbb{R}^n$ as in section 2.2:

$$f^{\text{SF}}(\lambda \cdot u) = f^{\text{SF}}(\lambda \cdot \left( (u^B)^T, (u^S)^T \right)^T) = \sqrt{x^T_{\lambda} R_{SF} x_{\lambda}} \overset{(\ast)}{=} \sqrt{(\lambda \cdot x)^T R_{SF} (\lambda \cdot x)}$$

$$= \lambda \cdot \sqrt{x^T R_{SF} x} = \lambda \cdot f^{\text{SF}}(u)$$

with

$$x_{\lambda} = \left( \text{Mkt}_{\text{int}} (\lambda \cdot u^B), \text{Mkt}_{\text{eq}} (\lambda \cdot u^S), \text{Mkt}_{\text{sp}} (\lambda \cdot u^B) \right)^T$$

$$x = \left( \text{Mkt}_{\text{int}} (u^B), \text{Mkt}_{\text{eq}} (u^S), \text{Mkt}_{\text{sp}} (u^B) \right)^T$$

The equality $(\ast)$ holds if and only if $\text{Mkt}_{\text{int}}, \text{Mkt}_{\text{eq}}$ and $\text{Mkt}_{\text{sp}}$ are also homogeneous of degree one. Firstly, it is

$$\text{Mkt}_{\text{int}} (\lambda \cdot u^B) = \max \left( \left( PV^{\text{int}} - PV^{\text{up}} \right)^T (\lambda \cdot u^B), \left( PV^{\text{int}} - PV^{\text{down}} \right)^T (\lambda \cdot u^B) \right)$$

$$= \lambda \cdot \max \left( \left( PV^{\text{int}} - PV^{\text{up}} \right)^T u^B, \left( PV^{\text{int}} - PV^{\text{down}} \right)^T u^B \right) = \lambda \text{Mkt}_{\text{int}} (u^B)$$

Secondly, it holds

$$\text{Mkt}_{\text{eq}} (\lambda \cdot u^S) = \sqrt{\bar{x}_{\lambda}^T R_{\text{eq}} \bar{x}_{\lambda}} \overset{(**)}{=} \sqrt{(\lambda \cdot \bar{x})^T R_{\text{eq}} (\lambda \cdot \bar{x})} = \lambda \cdot \text{Mkt}_{\text{eq}} (u^S)$$

for $\bar{x}_{\lambda} = \left( \text{Mkt}_{\text{eq, global}} (\lambda \cdot u^S), \text{Mkt}_{\text{eq, other}} (\lambda \cdot u^S) \right)^T$ and

$$\bar{x} = \left( \text{Mkt}_{\text{eq, global}} (u^S), \text{Mkt}_{\text{eq, other}} (u^S) \right)^T$$. To see $(**)$, we have to observe that

$$\text{Mkt}_{\text{eq, global}} (\lambda \cdot u^S) = \max \left( 0.3 \cdot \lambda \cdot \sum_{i=1}^{k_{\text{global}}} u^S_i, 0 \right) = \lambda \text{Mkt}_{\text{eq, global}} (u^S)$$

$$\text{Mkt}_{\text{eq, other}} (\lambda \cdot u^S) = \max \left( 0.4 \cdot \lambda \cdot \sum_{i=k_{\text{global}}+1}^{n_g} u^S_i, 0 \right) = \lambda \cdot \text{Mkt}_{\text{eq, other}} (u^S)$$

Thirdly, we can calculate

$$\text{Mkt}_{\text{sp}} (\lambda \cdot u^B) = \max \left( \text{stress}^T (\lambda \cdot u^B), 0 \right) = \lambda \cdot \max \left( \text{stress}^T u^B, 0 \right) = \lambda \cdot \text{Mkt}_{\text{sp}} (\lambda \cdot u^B)$$

Thus $f^{\text{SF}}$ is homogeneous of degree one.
A.6 Robustness check

In order to check the robustness of the presented approach to changes in the underlying $f^{true}$, the portfolio set-up is to be changed to coincide with the parameters presented in Eckert et al. (2016) and Gatzert and Martin (2012). On this basis, we can also consider an investment portfolio consisting of $n_B = 5$ bond investments, cf. Table 10, that are in line with Eckert et al. (2016) and $n_S = 2$ stocks, cf. Table 11. Here, the SCR employing the standard formula can be directly calculated. For the internal model, we again take into account correlated standard Wiener processes with correlation matrix $R_{IM}$ as discussed in section 2.1. The entries of $R_{IM}$ are presented in Table 12 and are again taken from Eckert et al. (2016). The stocks are then modeled as before following geometric Brownian motions with the parameters as in Table 11. Here, $S_1$, namely MSCI World, represents the investment in a “global” asset and $S_2$, India BSE 100, in an “other” asset. Again, we cover both potential classes in the equity risk sub-module. The parameters of the interest rate Vasicek (1977) process are given by $\kappa = 0.0953$, $\theta = 0.0437$ and $\zeta = 0.0069$, cf. Eckert et al. (2016)22. Furthermore, the initial value is set to $r(0) = \theta$.

### Table 10: Specifications of the bonds that are taken into account. The parameters are taken from Eckert et al. (2016). $B_1$: Colgate-Palmolive Company, $B_2$: Woolworth LTD, $B_3$: Areva SA, $B_4$: Poland Republic of (Government) and $B_5$: Turkey Republic of (Government).

<table>
<thead>
<tr>
<th>$B_i$</th>
<th>Type$_i$</th>
<th>rating$_i$</th>
<th>maturity$_i$</th>
<th>Coupon (in %)</th>
<th>$c_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Corporate</td>
<td>AA</td>
<td>10</td>
<td>2.950</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>Corporate</td>
<td>A</td>
<td>10</td>
<td>4.550</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>Corporate</td>
<td>BBB</td>
<td>11</td>
<td>3.500</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>Government</td>
<td>A</td>
<td>15</td>
<td>3.000</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>Government</td>
<td>BB</td>
<td>11</td>
<td>5.125</td>
<td></td>
</tr>
</tbody>
</table>

### Table 11: The parameters for the specification of the geometric Brownian motions representing stock investments. The numbers are taken from Gatzert and Martin (2012) who estimated them based on monthly data from 01/1988 to 07/2011.

<table>
<thead>
<tr>
<th>$S_i$</th>
<th>Index$_i$</th>
<th>rating$_i$</th>
<th>$\mu_i$</th>
<th>$\sigma_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>MSCI World</td>
<td>Global</td>
<td>0.0509</td>
<td>0.1574</td>
</tr>
<tr>
<td>2</td>
<td>India BSE 100</td>
<td>Other</td>
<td>0.1043</td>
<td>0.3309</td>
</tr>
</tbody>
</table>

22 These are based on the monthly “EURIBOR” data from 01/1999 to 12/2008.
The specifications of bond investments are presented in Table 10. Eckert et al. (2016) follow then Liang et al. (2011), assuming that bonds in the same rating class have the same parameters for their hazard rate Vasicek (1977) process which are shown in Table 13. Additionally, the recovery rate is set constant to $\delta_R = 0.61$ as before.

We evaluate the goodness of the approximation of $f^{true}(u)$ by $f^{SF}(u) + g_m(u)$ as suggested in section 2 for investing seven units into an equally weighted portfolio specified as $u_{initial} = (1, \ldots, 1)^T \in \mathbb{R}^n$. $f^{true}$ is evaluated on the basis of a Monte Carlo simulation with 5,000,000 paths and it results in capital requirements of $f^{true}(u_{initial}) = 1.525$ and $f^{SF}(u_{initial}) = 1.309$ resulting in a relative error of $-14.20\%$. Following the procedure described in section 2, seven scenarios can be identified which are presented in Table 14. Evaluating the goodness of the performance, we again take into account the shift presented in Eq. (18). Figure 4 presents the outcomes for $h \in [-1, 1]$. There, we obtain a similar result as in section 3. The standard formula is not capable of approximating the true risk reasonably underestimating the portfolio risk, and sensitivities are also not reflected correctly. An extension by $m = 1$ orthogonal convexity scenario overcomes one of

<table>
<thead>
<tr>
<th>$i$</th>
<th>$r$</th>
<th>$h_{AA}$</th>
<th>$h_A$</th>
<th>$h_{BB}$</th>
<th>$h_{BBB}$</th>
<th>$S_1$</th>
<th>$S_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r$</td>
<td>1</td>
<td>0.3</td>
<td>0.3</td>
<td>0.3</td>
<td>0.3</td>
<td>0.3</td>
<td>1</td>
</tr>
<tr>
<td>$h_{AA}$</td>
<td>0.3</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$h_A$</td>
<td>0.3</td>
<td>0.3</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$h_{BB}$</td>
<td>0.3</td>
<td>0.3</td>
<td>0.3</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$h_{BBB}$</td>
<td>0.3</td>
<td>0.3</td>
<td>0.3</td>
<td>0.3</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$S_1$</td>
<td>-0.26</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>$S_2$</td>
<td>-0.21</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.26</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 12: The entries of correlation matrix $R_{IM}$. The values are taken from Gatzert and Martin (2012) and Eckert et al. (2016) who have estimated the correlation between stocks and interest rate $r$ based on monthly data from 01/1988 to 07/2011. The correlations between bond classes $h_i$ and interest rate $r$ are originally from Liang et al. (2011).

<table>
<thead>
<tr>
<th>Rating</th>
<th>$\chi$</th>
<th>$\sigma$</th>
<th>$\Gamma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>AA</td>
<td>0.9581</td>
<td>0.0072</td>
<td>0.0181</td>
</tr>
<tr>
<td>A</td>
<td>0.7553</td>
<td>0.0141</td>
<td>0.0126</td>
</tr>
<tr>
<td>BBB</td>
<td>0.5865</td>
<td>0.0258</td>
<td>0.0113</td>
</tr>
<tr>
<td>BB</td>
<td>0.4406</td>
<td>0.0781</td>
<td>0.0454</td>
</tr>
</tbody>
</table>

Table 13: Parameters for the Vasicek (1977) processes modeling the hazard rates depending on the bond rating. The values are again taken from Eckert et al. (2016).
Table 14: Scenarios for the definition of the approximation function $g_m(u)$ in the setting of [Eckert et al. (2016)].

<table>
<thead>
<tr>
<th>i</th>
<th>Asset,</th>
<th>$x_{1i}^{OC}$</th>
<th>$x_{2i}^{OC}$</th>
<th>$x_{3i}^{OC}$</th>
<th>$x_{4i}^{OC}$</th>
<th>$x_{5i}^{OC}$</th>
<th>$x_{6i}^{OC}$</th>
<th>$x_{7i}^{OC}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$B_1$</td>
<td>0.0119</td>
<td>0.0115</td>
<td>-0.0147</td>
<td>0.0033</td>
<td>-0.0127</td>
<td>-0.0127</td>
<td>0.0056</td>
</tr>
<tr>
<td>2</td>
<td>$B_2$</td>
<td>-0.0044</td>
<td>0.0121</td>
<td>-0.0155</td>
<td>0.0021</td>
<td>-0.0137</td>
<td>-0.0137</td>
<td>0.0004</td>
</tr>
<tr>
<td>3</td>
<td>$B_3$</td>
<td>-0.0718</td>
<td>0.0174</td>
<td>-0.0239</td>
<td>-0.0010</td>
<td>-0.0213</td>
<td>-0.0213</td>
<td>0.0042</td>
</tr>
<tr>
<td>4</td>
<td>$B_4$</td>
<td>0.0557</td>
<td>0.0039</td>
<td>0.0009</td>
<td>0.0119</td>
<td>-0.0001</td>
<td>-0.0001</td>
<td>-0.0005</td>
</tr>
<tr>
<td>5</td>
<td>$B_5$</td>
<td>0.0534</td>
<td>0.0032</td>
<td>0.0018</td>
<td>-0.0163</td>
<td>-0.0072</td>
<td>-0.0072</td>
<td>-0.0005</td>
</tr>
<tr>
<td>6</td>
<td>$S_1$</td>
<td>-0.0100</td>
<td>0.0364</td>
<td>0.0424</td>
<td>0.0018</td>
<td>-0.0266</td>
<td>-0.0266</td>
<td>-0.0104</td>
</tr>
<tr>
<td>7</td>
<td>$S_2$</td>
<td>0.1814</td>
<td>-0.0845</td>
<td>0.0090</td>
<td>-0.0018</td>
<td>0.0815</td>
<td>0.0815</td>
<td>0.0013</td>
</tr>
</tbody>
</table>

these problems reflecting the true risk at $u_{\text{initial}}$ and also first order sensitivities correctly. Extending the regulatory risk calculation by $m = 2$ orthogonal convexity scenarios even allows for an inspection of non-marginal portfolio changes by reflecting the second order sensitivities of $f^{\text{true}}$ at least in some subspace. With these observations, we have seen

Figure 4: SCRs on the basis of the different risk measurement functions evaluating $u_{\text{new}}(h)$ as in Eq. (18) with the parameters in the internal model as reported in [Eckert et al. (2016)].

that the presented approach is robust to changes of the underlying true risk measurement function, making it applicable for a wide range of risk functions.
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